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MONEY IN THE THEORY OF ECONOMIC GROWTH: AN ANALYSIS OF COMPARATIVE-DYNAMIC AND OPTIMALITY ASPECTS OF GROWTH EQUILIBRIUM

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A Dissertation Presented

By

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PREFACE

The study of economic development or growth[1] has always been one of the central areas of economic study. The main thrust of Adam Smith's Wealth of Nations was to develop a theory of economic growth. Economic growth was also the primary concern of the great economists like Malthus, Marx, Schumpeter, and Keynes. Modern growth theory is built upon the foundations laid by these great masters.

R. F. Harrod, a pioneer of modern growth theory, was apparently inspired by John Maynard Keynes' insights into the problems of effective demand and investment-saving equilibrium in the determination of national income. As Joan Robinson states, "although Keynes' General Theory is strictly static in form, it has opened the way for a great outburst of analysis of dynamic problems."[2] In his essay, "Economic Possibilities for Our Grandchildren,"[3] Keynes expresses his concern with long-run economic growth, suggesting that the future rate of economic progress would depend primarily on four factors: (1) our power to control population; (2) our determination to avoid wars and civil dissensions;

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[1] Some economists distinguish between economic development and economic growth, the latter including only economic variables and the former embracing socio-political, cultural, and institutional as well as conventional economic variables. I feel, however, that the socio-political, cultural, and institutional variables will ultimately have to be incorporated into the theory of economic growth and hence do not distinguish between these two terms.

[2] Joan Robinson, The Rate of Interest and Other Essays (London: MacMillan, 1952), Preface.

[3] John Maynard Keynes, Essays in Persuasion (London: Rupert Hart-Davis, 1952), p. 373.

(3) our willingness to entrust to science the direction of those matters which are properly the concern of science; and (4) the rate of accumulation as fixed by the margin between our production and our consumption.

The early pioneering works of Harrod (1939) and Domar (1946) are the extensions of Keynesian economics. The Harrod-Domar model demonstrated that the capitalistic economy is highly unstable not only in the short run but also over the long run. The implication is that long-run steady-state growth is possible only under very unlikely circumstances, i.e., only when the warranted rate of growth is equal to the natural rate in Harrod's term. Therefore, public policies designed to achieve a stable growth are required.

The Harrod-Domar model has been criticized on the grounds that the model is unrealistically rigid and unstable. A number of economists including Solow, Swan, and Kaldor attempted to reach more flexible conclusions by relaxing or changing some of the assumptions made by Harrod and Domar. Solow and Swan assume a continuous neoclassical production function rather than assuming fixed factor-proportions in production. Kaldor introduces a differential savings function rather than assuming a constant overall savings ratio for the entire economy.[1]

Whether of the neoclassical or the Harrod-Domar type, these models are incomplete because they do not incorporate monetary variables. James Tobin was one of the first economists[2] who had realized the importance of monetary factors in growth theory. Tobin states:

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[1]N. Kaldor and J. A. Mirrlees, "A New Model of Economic Growth," Review of Economic Studies, XXIX (June, 1962).

[2]Others include John G. Gurley, Edward S. Shaw, and Alain C. Enthoven.

> In non-monetary neo-classical growth models, the equilibrium degree of capital intensity and correspondingly the equilibrium marginal productivity of capital and the rate of interest are determined by "productivity and thrift," i.e., by technology and saving behavior. Keynesian difficulties, associated with divergence between warranted and natural rates of growth, arise when capital intensity is limited by the unwillingness of investors to acquire capital at unattractively low rates of return. But why should the community wish to save when rates of return are too unattractive to invest? This can be rationalized only if there are stores of value other than capital, with whose rates of return the marginal productivity of capital must compete.[1]

Indeed, Tobin's question, "why should the community wish to save when rates of return are too unattractive to invest?" enabled the economists to escape the unrealistic neoclassical world of "real variables" in which real saving must inevitably equal real investment.

Since Tobin's first work appeared in 1955,[2] there has been a great deal of research in the field of monetary growth theory. Chapter I provides a review of the monetary growth models of various kinds. This chapter is designed to facilitate understanding of subsequent chapters. Models by Tobin, Levhari and Patinkin, Sidrauski, and Stein are used for this purpose.

The present writer is convinced that the fundamental questions to be asked concerning the monetary growth theory are: (1) Can the variations in the rate of monetary expansion influence the time profiles of the real variables and therefore the equilibrium values of real variables? (2) Is there an optimal rate of growth of the supply of money?

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[1] James Tobin, "Money and Economic Growth," Econometrica, 33 (October 1965), p. 671.

[2] James Tobin, "A Dynamic Aggregative Model," Journal of Political Economy, 63 (April, 1955).

(3) Is there an optimal degree of financial intermediation? The first question concerns the problem of neutrality of money in the growth theory context and hence the examination of the comparative-dynamic characteristics of the model. There are currently three hypotheses concerning this question. The first and most widely accepted hypothesis has been suggested by Tobin: That an increase in the rate of monetary expansion will ultimately result in an increase in the equilibrium degree of capital intensity and a decrease in the equilibrium value of per capita real balances. The second hypothesis proposed by Sidrauski states that the long-run equilibrium capital intensity is not affected by variations in the rate of monetary expansion although in the short run the rate of capital accumulation may fall as a result of an increase in the rate of monetary expansion. The third hypothesis (Stein) simply states that the equilibrium capital intensity may either increase, remain constant, or fall as a result of an increase in the rate of monetary expansion.

The second question is concerned with the problem of optimality. There is no definite conclusion on this topic at the present time, but Tobin states:

> ...there is an optimal rate of growth of the supply of outside money, equal to or smaller than the nominal interest rate i, only to the extent that diversion of saving into this vehicle is necessary to keep the marginal productivity of capital from falling below n (the natural rate). In the absence of such a tendency for oversaving, it is not optimal to absorb any saving in outside money or deadweight debt.[1]

[1]James Tobin, "Notes on Optimal Monetary Growth," JPE, 76 (July/August, 1968), p. 841.

On the other hand, Levhari and Patinkin conclude that the Golden Rule is not valid in the monetary model and the optimal rate of growth of non-interest-bearing outside money is zero. However, they do not consider the case in which money is allowed to bear nominal interest.

As far as the third question is concerned, no satisfactory inquiry has so far been made except Tobin's discussion on money as a means of payment and on the optimum stock of means of payment.[1] The present writer has made no attempt to discuss the third question in depth, not because he considers it the least important but because answering this question at the present stage of development of economic theory is considered beyond his intellectual ability.

Instead, this dissertation is concentrated on the first two questions, i.e., the comparative-dynamic and optimality aspects of growth equilibrium. The whole analysis is concerned with the state of growth equilibrium --- the so-called steady state. Particularly, this dissertation concentrates on different steady-state paths associated with different rate of monetary expansion or different rates of price change and on the conditions ensuring the optimum steady-state paths.

Therefore, the analysis is based upon a differential equations structure representing a dynamic system of the model economy. The model economy is an advanced market economy with a sophisticated banking system and a well-developed capital market. The price level and capital stock are assumed to be given by the past.[2] The community can choose the rate

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[1] Ibid., pp. 843-59.

[2] All variables are expressed in per capita terms.

of monetary expansion (and hence the rate of price change). The rate of capital accumulation is determined once the rate of price change and the saving hypothesis (or the type of investment function) have been choosen.[1] Thus, these choices determine the new values of the price level, the stock of capital, and the stock of monetary assets for the next time period via the differential equations system. The process is repeated.

Both equilibrium and disequilibrium methods are used. The equilibrium model is discussed in Chapters III and V where it is assumed that there is no independently determined investment function but rather that all saving plans are realized. In the equilibrium model all markets are assumed to be in equilibrium and hence all productive factors are assumed to be fully employed. A disequilibrium model is introduced in Chapter IV, where it is assumed that both money and commodity markets are normally out of equilibrium and a disequilibrium in these markets activates price changes and therefore a divergence between actual and planned investments. Both equilibrium and disequilibrium models represent dynamic systems of the model economy introduced in Chapter II. These models contain a number of modifications of conventional monetary growth models.

First, the models presented, whether equilibrium or disequilibrium, are characterized by: (1) The bond market is introduced in addition to the money market;[2] (2) Both money and bonds, inside and outside, are considered a part of the community's wealth and also factors of production. Therefore, a new kind of production function and a new definition of real disposable income are used in the models presented in this thesis.

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[1]In disequilibrium model, an investment function is introduced instead of saving hypothesis.

[2]The Stein model contains a bond market but only outside bonds are considered a part of community's wealth and neither real balances nor real bonds are considered factors of production.

Second, the models appearing here modify the conclusions made by other writers significantly: In the equilibrium model it has been shown that an increase in the rate of monetary expansion results in a rise in the equilibrium capital intensity and a fall in the equilibrium values of per capita real balances and real bonds (bonds in real terms) so long as a small amount increase in one factor of production increases the marginal productivities of the remaining factors at the same rate.[1] In the equilibrium model it has also been shown that the optimal rate of growth of money is zero regardless of whether money is at its satiety level or not when money is not allowed to bear nominal interest, but that the optimal rate of growth of money is equal to the nominal rate of interest on money itself when nominal interest is paid on money, provided that the fund needed for the payment of interest on money is created by the banking system without cost. If such a fund comes from the increased productivity of capital made possible by the increase in the stock of real balances, the optimal rate of monetary expansion will be twice the nominal rate of interest on money.

Finally, in the disequilibrium model it has been shown that the effects of variations in the rate of monetary expansion on the equilibrium values of real variables are ambiguous unless the exact nature of the current institutional market arrangements determining how output is divided between investors and savers in times of rising prices and the speed of market response in eliminating excess demands are known.

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[1]This result supports Tobin's hypothesis that an accelerated decline in prices will retard the flow of savings into capital formation.

In short, the present writer has made an attempt in this dissertation to throw some lights on the theory of monetary economic growth by introducing a model which contains a four-factor production function and treats both money and bonds, inside and outside, as a part of community's wealth. A number of significant results listed above have been obtained concerning the comparative-dynamic and optimality aspects of growth equilibrium.

The present writer gratefully acknowledges the assistance given in completing this study by the members of his graduate committee and by his colleagues and friends. Although many of his fellow graduate colleagues have contributed to this study, the present writer must single out and thank Mr. Kimon Constas, who has spent much time discussing thorny theoretical problems with him.

My graduate committee, under the chairmanship of Professor Robert M. Lovejoy, deserves special acknowledgement. Professor Lovejoy has been keenly interested in this study and made a number of valuable comments and criticisms to improve this writing. His particular interest in the field of monetary growth theory and his confidence in me were a major factor in stimulating my work.

A special word of appreciation is due to Professor John E. La Tourette, the Chairman of the Economics Department, for his constant encouragement and guidance.

Professor Kenneth K. Kurihara, a member of the Committee, deserves special acknowledgement. Only those who have been fortunate enough to have worked under his guidance can possibly understand the nature of my debt to him. It was while a student of Professor Kurihara's in his courses and seminars both at Rutgers University and at the SUNY at Binghamton that I became interested in the growth theory and it was his own important contributions in this field that inspired this writing. In this connection, I must acknowledge a constant encouragement given by Professor Robert J. Carlson, who directed my writing of M.A. thesis while at the University of South Carolina and who has been expressing his interest and confidence in my graduate work both at Rutgers University and the SUNY at Binghamton.

I am grateful to my parents who have given me a strong moral support and constant encouragement which I needed throughout the course of my graduate study. A special word of appreciation goes to my wife, Kyoung-Ja, for her patience and willingness to share in the opportunity cost of this writing. Finally, I must acknowledge the excellent typing assistance given by Mrs. Judith Falkins. Needless to say, without the help of all these people, the thesis would not have seen the light. Nevertheless, I alone am responsible for possible errors and defects in this study.

J. S. Y.

Binghamton, New York
August, 1971

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CHAPTER I

MONEY IN THE THEORY OF ECONOMIC GROWTH:
AN INTRODUCTION

The purpose of this introductory chapter is two-fold. Firstly, it is to indicate what the central questions in the theory of monetary economic growth are, and how we are going to deal with them. Secondly, it is to review the current literature with regard to these questions.

Methodology

It is to be emphasized that the thesis is concerned with the state of long-run growth equilibrium or steady state. Therefore, it abstracts from the short-run adjustments process and concentrates on the various aspects of steady state. Our approach is one of model building. Various simplifying assumptions are made and the relationships among endogenous variables are established under these assumptions in such a way that makes the system capable of yielding a solution. All this is done not because we want to ignore the details of economic phenomena and economic reality but because we want to abstract from unnecessary details, ignore irrelevant points and concentrate on the matters which are of our immediate concern. The thesis is aiming at clarifying some of the central issues in the monetary growth theory and at shedding some lights on the central questions in the field.

For this purpose we use both equilibrium and disequilibrium methods. In equilibrium approach it is assumed that all markets are in equilibrium or that in all markets the equilibrium is immediately restored by perfectly functioning market mechanism which eliminates any kind of excess demand instantaneously. In disequilibrium approach there always exists a possibility of having a disequilibrium in any of these markets. However, it is assumed that the equilibrium will eventually be restored in the long run but after a considerable length of adjustment period. Especially commodity and money markets are assumed to be highly vulnerable to disequilibrium.

Whether we use the equilibrium or disequilibrium method, our analysis is based on a differential equations structure since we are primarily interested in the dynamic system which describes the time paths of real variables. We take a price level and capital stock to be given by the past.[1] Then the time paths of capital-labor ratio and per capita real balances and real bonds are determined by the rate of price change and the saving hypothesis. Once the rate of price change (or equivalently the rate of monetary expansion) is chosen along with the saving hypothesis, a system of differential equations determines the stock of capital and the price level, and hence the capital intensity and per capital real value of monetary assets, for next instant. This process is repeated. If we assume a constant saving ratio, the rate of price change becomes the only policy parameter. Then a question to be asked is whether variations in the rate of monetary expansion and hence variations in the rate of price change will affect the time paths and equilibrium values of real variables and, if so, in what direction. This is a question of comparative dynamics in the sense

[1]All variables may be expressed in per-capita terms.

that the variations in the rate of monetary expansion will, if money is not neutral in the growth context, cause the economy to settle down on a different steady-state path with a different set of efficiency wage rate and rental rate.

Another important question in the growth economics is to find a time path associated with a maximum utility — a question of optimum growth. Since the government can choose only one variable, the rate of monetary expansion, as a policy parameter under the assumption of constant saving ratio, our question is to determine what the optimal rate of monetary expansion is.

The questions of comparative dynamics and optimum growth are discussed under the given institutional framework. But there is also a question of changing the financial institutions themselves in order to increase the efficiency of the system. This is a question of optimal degree of financial intermediation which is not discussed in depth in the thesis. In the thesis we concern ourselves primarily with the first two of the three central questions in the monetary economic growth theory, i.e., the comparative-dynamic and optimality aspects of the growth equilibrium.

Many attempts have been made by many economists to obtain satisfactory results on these questions. However, most of the monetary growth models appeared so far contain deficiencies primarily due to the fact that they are all outside-money models[1] and that they exclude the bonds market. I have tried in this thesis to build a model which includes bonds market in addition to money market and which has a production function in which capital, labor, real balances and real bonds all appear as factors of production. In this model both inside and outside assets are considered a part of community's wealth and, therefore, the old definition of real disposable income is modified accordingly. The wealth is defined as an asset generating future income streams. The question of how inside assets can be considered as wealth is considered in detail in Chapter II. This new model is introduced in Chapter II and its dynamic system is developed in Chapter III where equilibrium method is used and also in Chapter IV where disequilibrium method is used. Chapter V is concerned with the question of optimum growth. The results obtained are significantly different from those obtained in other monetary growth models. These results are summarized in Chapter VI.

In what follows we take the models of Tobin, Levhari-Patinkin, Sidrauski and Stein as the representative among many monetary growth models and review them with regard to the three central questions: (1) Can variations in the rate of monetary expansion influence the time profiles and equilibrium values of real variables? (2) Is there an optimum rate of growth of money? (3) Is there an optimal degree of financial intermediation?

[1]Outside money is defined as non-interest-bearing government debt and generally considered an addition to the community's wealth, since no real resources are required to create such a debt. Inside money is created by the banking system in the form of debts and claims such as demand deposits within the private sector. Inside money is not usually considered an addition to the community's wealth on the grounds that it simply represents debts and claims which can be cancelled out within the private sector. However, Pesek and Saving argue that inside money is also a form of wealth as long as it remains as debts and claims and the solvency of the banking system is maintained. For detailed discussions, see Chapter II.

The Equilibrium Capital Intensity in Monetary and Non-Monetary Growth Models

James Tobin was the first one who realized the important fact that the Keynesian concept of liquidity trap — indeed any idea of divergence between saving and investment — presupposes that there exist alternative stores of value besides real capital. According to Tobin, Keynesian difficulties associated with divergence between warranted and natural rates of growth arise when investment is limited by unattractively low rates of return. The excess of saving over investment can be rationalized "only if there are stores of value other than capital, with whose rates of return the marginal productivity of capital must compete."[1] The alternative stores of value can be money, bonds, or any kind of financial assets. When there exist various forms of monetary assets, there is no reason why all the savings must be held exclusively in the form of real capital. Part of the saving is likely to be held in the form of monetary assets for various reasons. The portfolio decision will be based upon such factors as the rates of return on various forms of assets, risk and uncertainty. Then the Solow-type growth models based upon the assumption that capital formation is determined solely by the propensity to save of the society is no longer valid. In the monetary economy the propensity to save determines how much is saved rather than consumed but does not determine in what forms savers hold their savings. This point is made clear by Tobin who states:[2]

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[1] James Tobin, "Money and Economic Growth," Econometrica, 33 (October, 1965), p. 671.

[2] Ibid.

> Fisher and Keynes, among others, have drawn the useful and fruitful analytical distinction between choices affecting the disposition of income and choices affecting the disposition of wealth. The first set of choices determine how much is saved rather than consumed and how much wealth is accumulated. The second set determines in what forms savers hold their savings, old as well as new.

This distinction between two sets of choices is a fundamental departure from the Solow-type conventional neoclassical growth model, since it makes the time path of capital accumulation in the monetary model different from that in the non-monetary model. In Tobin's monetary growth model which assumes that outside money which, is the only type of monetary asset, is injected into the economy by means of government transfer payment, the time path of capital accumulation is represented by a differential equation[1]

(1.1) $\dot{k} = sy(k) - (1 - s)(\mu - \pi)m - nk$

where k = capital-labor ratio ($\dot{k} = dk/dt$), y = per capita output, m = per capita real balances, s = savings ratio (constant), n = rate of growth of effective labor (exogenously determined), μ = rate of growth of nominal stock of money, and π = rate of price change. Equation (1.1) compares with Solow's differential equation[2]

(1.2) $\dot{k} = sy(k) - nk.$

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[1]For mathematical derivation, see Appendix II.

[2]See Appendix I.

The comparison of (1.1) and (1.2) reveals the fact that the equilibrium capital intensity in Tobin model is lower than that in Solow model. The equilibrium capital intensities represented by k^* are

$$(1.3)\quad k^* = sy(k)/n - (1 - s)m$$

in Tobin model and

$$(1.4)\quad k^* = sy(k)/n$$

in Solow model.[1] Many economists were puzzled with the idea that the equilibrium capital intensity is lower and hence per capita output and consumption are lower in the monetary model because the introduction of money into the model seemed to cause the economy to settle down on a growth path which is associated with lower per capita output and consumption. Levhari and Patinkin ask, "... if the sole result of introducing money into an economy were to reduce k and hence per capita output and consumption, why should it be introduced? Where are the vaunted advantages of a monetary economy?"[2] However, the fact that the equilibrium capital intensity is lower in the monetary "model" does not necessarily mean that the equilibrium capital intensity in a monetary "economy" is lower than what would have been in the barter "economy." The Solow-type non-monetary models are not "barter" models. They are "non-monetary" models in the sense that they are the

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[1]The equilibrium capital intensity can be found by setting $\dot{k} = 0$, since in growth equilibrium (steady state) all variables including K and L grow at the same rate and hence k is constant.

[2]David Levhari and Don Patinkin, "The Role of Money in a Simple Growth Model," American Economic Review, LVIII (September, 1968), p. 717.

models of monetary economy but fail to take the monetary factors into account. This point has been made clear and emphasized by Frank Hahn who states:[1]

> Present growth models which make no allowance for money are not barter models. No attention is paid to the economics of transactions and one must suppose that in this world the "mediating" function of money is performed costlessly by some outside agency.

Therefore, the question raised by Levhari and Patinkin must be considered inappropriate.

The main goal of Tobin's monetary growth model is, of course, not to demonstrate that the equilibrium capital intensity is lower in the monetary model but to indicate the possibility that there might be a way out of the so-called "Harrodian impasse" --- the divergence between the warranted and natural rates of growth --- through the appropriate monetary policies, since the equilibrium values of real variables are affected by the variations in the rate of monetary expansion, i.e., money is not neutral in the growth context. The Harrodian impasse is well described by Tobin as follows:[2]

> Harrod, for example, argues that investors will simply not undertake new investment unless they expect to receive a certain minimum rate of return. Savers, on the other hand, are not discouraged from trying to save when yields fall to or below this minimum. The result is an impasse which leads to Keynesian difficulties of deficit demand and unemployment. In Harrod's model these difficulties arise when the warranted rate of growth at the

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[1]Frank Hahn, "On Money and Growth," Journal of Money, Credit and Banking, I (May, 1969), p. 172.

[2]Tobin, op. cit., p. 675

> minimum required rate of profit exceeds the natural rate. The rate of saving from full employment output would cause capital to accumulate faster than the labor force is growing. Consequently, the marginal product of capital would fall and push the rate of return on investment below the required minimum.

However, the fact that there exists a required minimal rate of return on capital reflects "the competition of other channels for the placement of saving" or the existence of alternative stores of value, i.e., monetary assets. Suppose that the warranted rate at the current rate of profit is higher than the natural rate due to excessive saving and the rate of profit is already so low that investors are not willing to undertake investment any further. This will lead to the Harrodian impasse if all the saving must take the form of capital. But if there exist alternative stores of value which can absorb part of community's excessive saving, all will be well.

Tobin suggests that there are two ways in which government policy can avoid this impasse. One measure the government could take is to reduce the yield on financial assets so that investment in real capital may become more attractive. Alternatively, the government could channel part of community's excessive saving into increased holdings of financial assets. This will reduce the rate of capital accumulation and thus equilibrium capital intensity. This can be done, for example, by means of accelerating the rate of deflation. "The accelerated decline in prices, by augmenting the real value of existing money balances, helps to restore portfolio balance. Moreover, by increasing total real wealth it retards the flow of saving into capital formation.'[1] This is a significant conclusion because according to this theory a decrease (increase) in the rate of monetary expansion will decrease (increase) the equilibrium capital intensity by decreasing (increasing) the rate of price change. Therefore, the equilibrium capital intensity can be manipulated and set at such a level that is consistent with maintaining the rate of profit at the required minimum. In short, what Tobin tries to show is that "the equilibrium interest rate and degree of capital intensity are in general affected by monetary supplies and portfolio behavior, as well as by technology and thrift,"[2] and thus the Harrodian impasse can be removed by the appropriate monetary policies.

On the other hand, Levhari and Patinkin devote much of their effort to the demonstration of the fact that the equilibrium capital intensity in the monetary model can be higher than that in the non-monetary model. Like Tobin, they make the assumption that there is only one type of monetary asset, outside money, and that the government injects outside money into the economy by means of transfer payment and withdraws by means of taxes.[3] If transfer payment is financed by the same amount of tax receipts, the stock of outside money will not increase. Levhari and Patinkin maintain that a rationale for the positive demand for real money balances must

[1] Ibid., pp. 682-3.

[2] Ibid., p. 684.

[3] Levhari and Patinkin make a further assumption that money is non-interest-bearing whereas Tobin allows outside money to bear a nominal rate of interest.

interpret money balances either as a consumer's good or as a producer's good. In the consumer's good approach the services rendered by money balances appear in the individual's utility function and hence in his real disposable income while in the producer's good approach such services reflect themselves in the production function.

In the consumer's good approach, the services of real balances are valued at the alternative cost at the margin of holding money balances which is equal to Fisher's money rate of interest, $r + \pi$, where r is the real rate of interest equal to the marginal productivity of capital, and thus the imputed services are included in real disposable income. Thus the relevant definition of disposable income is, according to Levhari and Patinkin,

$$(1.5)\quad Y^* = Y + (\mu - \pi)M/P + (r + \pi)M/P = Y + (\mu + r)M/P$$

where Y^* = disposable income, Y = physical output, M = nominal stock of money, and P = general price level.[1] This is rather a strange definition of disposable income because it is not, according to this definition, affected by price changes. Levhari and Patinkin explain: "The decrease in the real value of real balances caused by a price increase, $\pi(M/P)$, does not appear as a deduction from disposable income; for it is offset by the fact that $\pi(M/P)$ also represents part of the imputed income from the holding of real balances."[2]

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[1]Both physical output and disposable income are measured in the number of units of a single commodity which can be either consumer's good or producer's good.

[2]Levhari and Patinkin, op. cit., p. 718.

Assuming that a constant proportion, s, of disposable income is saved and that "the price level instantaneously adjusts itself so as to equate the demand and supply for real balances,"[1] Levhari and Patinkin derive the following differential equation:[2]

$$(1.6)\quad \dot{k} = sy(k) - (1 - s)(\mu - \pi)m + s(r + \pi)m - nk$$

which compares with Solow's equation (1.2) and Tobin's equation (1.1). Assuming further that the demand for real balances is an increasing function of physical output such that $M/P = \rho Y(K,L)$, i.e., $m = \rho y(k)$, where ρ is a factor of proportionality,[3] (1.6) is rewritten as

$$(1.7)\quad \dot{k} = [s(1 - s)(\mu - \pi)\rho + s(r + \pi)\rho]y(k) - nk,$$

Then the equilibrium capital intensity which can be derived from (1.7) by setting $\dot{k} = 0$ is

$$(1.8)\quad k^* = y(k)/n[\{s\ 1 + \rho(n + \pi + r)] - \rho n\} = \sigma y(k)/n$$

where $\sigma = s[1 + \rho(n + \pi + r)] - \rho n$ is a "physical" savings ratio which is a ratio of savings devoted to real capital formation to physical output.

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[1] Ibid., p. 719.

[2] For mathematical derivation, see Appendix III(a).

[3] ρ may be interpreted as an income velocity. It is assumed to be a function of money rate of interest, $r + \pi$. The function $M/P = \rho Y(K,L)$ is assumed to be homogeneous of degree one.

Now we can compare the equilibrium capital intensity in the Solow-type non-monetary model represented by (1.4) and the equilibrium capital intensity in Levhari-Patinkin monetary model represented by (1.8). The only difference between these two is that the "physical" savings ratio, σ, appears in the latter instead of the "overall" savings ratio, s. Therefore, it can be concluded that the equilibrium capital intensity in the monetary model will be higher (lower) than that in the non-monetary model if the physical savings ratio is greater (smaller) than the overall savings ratio, that is, if the fraction of physical output which is devoted to real capital formation is greater (smaller) than the fraction of disposable income which is devoted to overall saving or accumulation of wealth.

When money is alternatively treated as a producer's good, the relevant production function has the form $Y = Y(K,L,M/P)$ or $y = y(k,m)$ in per capita terms, if Y is linear homogeneous. In this approach the imputed services of money balances are not included in disposable income, since these services are reflected in the increased output of commodities which these money balances make possible. Hence the definition of disposable income in this approach is the same as Tobin's definition except that the physical output in Tobin model is a function only of real capital. Also in this approach we have a different demand function for real balances, since the treatment of real balances as a factor of production implies that the demand for them is determined by the marginal productivity principle.

Again assuming a constant savings ratio, the following differential equation describing the time path of k is derived:[1]

[1]For mathematical derivation, see Appendix III(b).

$$(1.9)\quad \dot{k} = sy(k,m) - (1 - s)(\mu - \pi)m - nk$$

from which the equilibrium capital intensity k^* is derived so that

$$(1.10)\quad k^* = [s - (1 - s)mn/y(k,m)]\ y(k,m)/n = \sigma y(k,m)/n$$

where $\sigma = s - (1 - s)mn/y(k,m)$ is a "physical" savings ratio. From the factor market equilibrium condition, $y_k(k,m) = y_m(k,m) - \pi$, which is implied by the marginal productivity principle, it follows that m is a function of k and π, i.e., $m = m(k,\pi)$. Therefore, the physical savings ratio will also be a function of k and π. In other words, the physical savings ratio can be manipulated through monetary policies varying the rate of monetary expansion and hence the rate of price change. The variations in the rate of monetary expansion will influence the equilibrium capital intensity and the equilibrium per capita real balances via its effect on the rate of price change. Therefore, the equilibrium capital intensity can be higher in the monetary model if the physical savings ratio becomes greater than the overall savings ratio as a result of introducing money into the model. Ultimately the equilibrium capital intensity is a function of the rate of price change or the rate of monetary expansion. Thus one of the fundamental questions in the monetary growth theory is to determine what effects the changes in the rate of monetary expansion will have on the equilibrium capital intensity and equilibrium per capita real balances, which is a question of comparative dynamics. In other words, one of our primary tasks is to determine the signs of $dk/d\mu$ and $dm/d\mu$, or equivalently $dk/d\pi$ and $dm/d\pi$.[1]

[1]It follows from $m = M/(PL)$ that $\dot{m}/m = \mu - \pi - n$. In the steady state, $\dot{m} = 0$, i.e., $\mu - \pi - n = 0$. Taking differentials yields $d\mu - d\pi = 0$. Hence $d\mu = d\pi$.

The Comparative-Dynamic Properties of Growth Equilibrium

In the previous section it was shown that Tobin's primary concern was to demonstrate the fact that "the equilibrium interest rate and degree of capital intensity are in general affected by monetary supplies and portfolio behavior, as well as technology and thrift,"[1] and hence the Harrodian impasse can be removed by appropriate monetary policies. The implications of this statement can best be demonstrated by the use of Figure 1-1.[2]

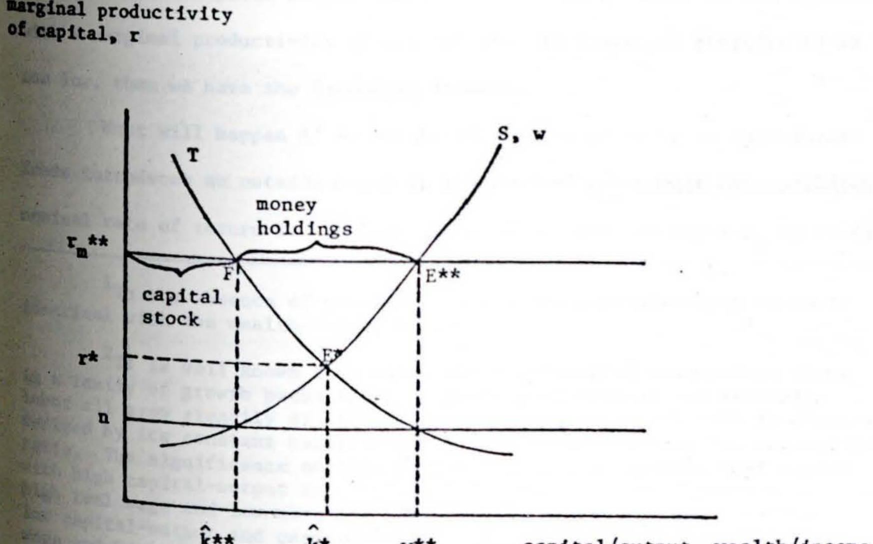

Figure 1-1. Determination of Equilibrium Capital-Output and Wealth-Income Ratios

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[1] Tobin, op, cit., p. 684.

[2] Figure 1-1 is a reproduction of Figure 1 in Tobin, "Notes on Optimal Monetary Growth," Journal of Political Economy, 74, No. 4, Part II (July/August, 1968), p. 835.

Figure 1-1 relates the capital-output ratio (or wealth-income ratio)[1] to the marginal productivity of capital for alternative steady-state growth paths.[2] "Curve T represents the technological relationship between these variables implicit in the economy's production function. Paths with higher capital-output ratios will have lower marginal productivities of capital."[3] Curve S represents "the amount of wealth savers desire, relative to national income, in a situation of balanced growth."[4] It demonstrates the fact that "savers may desire a higher wealth-income ratio along a path with a higher real rate of interest than along one where the reward for saving is low."[5] The equilibrium capital-output ratio is therefore $\hat{k}^*$ which is associated with a marginal productivity of capital r^*. If investors consider r^* as too low, then we have the Harrodian impasse.

What will happen if an alternative store of value is introduced? Tobin introduces an outside money in the form of government debt yielding nominal rate of return i. The real rate of interest on money r_m is equal

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[1]In the absence of monetary assets, the capital-output ratio is identical with the wealth-income ratio.

[2]It is well known that under the neoclassical assumptions there is a family of growth paths along which output, capital and effective labor all grow steadily at the natural rate and that each path is characterized by its constant capital-output ratio and thus constant capital-labor ratio. The significance of this proposition is, of course, that a path with high capital-output and capital-labor ratios is associated with high real wage and low marginal productivity of capital and a path with low capital-output and capital-labor ratios is associated with low real wage and high marginal productivity of capital.

[3]Tobin, "Notes on Optimal Monetary Growth," p. 834.

[4]Ibid.

[5]Ibid.

to $i - \pi$ or $i - \mu + n$ in the steady state.[1] If an assumption is made that money and capital are perfect substitutes in the portfolios of savers, it is required, for both capital and money to exist simultaneously, that

(1.11) $r = r_m = i - \mu + n.$

The significance of (1.11) is that the government can, by manipulating i and μ, and hence r or r_m, manipulate the equilibrium capital-output ratio. For example, in Figure 1-1 the government can, by selecting r_m^{**}, determine the equilibrium capital-output ratio as $\hat{k}^{**}$.

However, with an alternative store of value, the capital-output ratio is no longer the same as the wealth-income ratio. Since the total wealth is now $K + M/P$ and the disposable income is $Y + D(M/P)$, the wealth-income ratio is now related to the capital-output ratio in the following way:

(1.12) $$\hat{k} = \frac{w}{1 + \hat{m}(1 - nw)}$$

where w is the wealth-income ratio and $\hat{m}$ is the ratio of real balances to capital.[2] In (1.12), nw may be considered a ratio of total saving to disposable income (i.e., overall saving ratio) in the same sense that $n\hat{k}$

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[1]In the steady state, $n = \mu - \pi$.

[2]$w = \dfrac{K + M/P}{Y + D(M/P)}$ by definition. In the steady state, $D(M/P) = (\mu - \pi)M/P = nM/P$ so that $w = (K + M/P)/(Y + nM/P) = K(1 + \hat{m})/(Y + n\hat{m}K) = \hat{k}(1 + \hat{m})/(1 + n\hat{m}\hat{k})$. Therefore, $\hat{k}(1 + \hat{m}) = w(1 + n\hat{m}\hat{k})$, which implies (1.12).

represents a constant saving ratio in an economy where there is no monetary asset.[1] Since nw will be smaller than one, the capital-output ratio $\hat{k}$ is smaller than, equal to, or larger than the wealth-income ratio w according as $\hat{m}$ is positive, zero, or negative. Retaining the assumption that money and capital are perfect substitutes,[2] the Curve S in Figure 1-1 can still represent the relationship between desired wealth-income ratio and marginal productivity of capital. Therefore, when the real rate of interest is set at r_m**, the desired wealth-income ratio is given by Curve S as w**. In Figure 1-1, w** is greater than $\hat{k}$**. This means that $\hat{m}$ is positive according to (1.12). This represents an equilibrium situation since the point ($\hat{k}$**, r_m**) is on Curve T and the point (w**, r_m**) is on Curve S respectively. The distance FE** is the society's money holdings as a ratio of disposable income. The government can always, by selecting the values of i and μ appropriately, determine the equilibrium capital-output and wealth-income ratios. Thus the Harrodian difficulties can be removed. A decrease in the real rate of interest on money lowers the equilibrium wealth-income ratio while increasing the equilibrium capital-output ratio. Hence, given the nominal rate of interest i, an increase in the rate of

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[1] In an economy where there is no monetary asset but real capital is the only kind of asset, a growth equilibrium is characterized by $n = s/\hat{k}$ or $s = n\hat{k}$.

[2] This assumption appears to be the major weakness of Tobin model.

monetary expansion μ will increase the equilibrium capital-output and capital-labor ratios. In other words, the sign of comparative-dynamic derivative $dk/d\mu$ will be positive.[1]

Harry G. Johnson takes Tobin's conclusion as an indication of non-neutrality of money in the growth context and contends that this result is due to the fact that Tobin model is outside-money model in which "real balances constitute an addition to material wealth in the form of capital goods."[2] Johnson argues that if it were assumed that money is of the inside variety, created against private debts, in which case "real balances would not constitute a net addition to material wealth but instead an indirect means of holding material wealth, ... monetary policy would not be able to influence growth through its influence on the magnitude of the supplement to earned income received in the form of additional real balances, and therefore on the material

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[1]Tobin realizes that there are two opposing forces involved here, namely, the Pigou effect and the Wicksell effect. But he assumes that the Pigou effect working in favor of his proposition will eventually win out. He states: "Evidently there are two effects, at war with each other. One we might call the Pigou effect, the other the Wicksell effect. The Pigou effect is stabilizing. Consider the case of a deflationary shock. The accelerated decline in prices, by augmenting the real value of existing money balances, helps to restore portfolio balance. Moreover, by increasing total real wealth it retards the flow of saving into capital formation. The Wicksell effect is destabilizing. An accelerated decline in prices means a more attractive yield on money and encourages a further shift in portfolio demand in the same direction as the original shock. There is no a priori reason why one effect should be stronger than the other in the neighborhood of equilibrium. In the model under discussion, the Pigou effect will eventually win out, but only after what may be a prolonged period of deflation, zero or negative capital formation, and retarded growth." See Tobin, "Money and Economic Growth," pp. 682-3.

[2]Harry G. Johnson, "Money in a Neo-Classical One-Sector Growth Model," Essays in Monetary Economics (Harvard University Press, 1967), Ch. IV, p. 117.

savings ratio,"[1] but instead "the influence of monetary policy on growth would be confined to the influence of the target rate of inflation or deflation on the utility yield of real balances."[2] However, if the legal restriction of non-payment of interest on demand deposits were removed, "competition in the commercial banking business would result in holders of demand deposits being offered a rate of interest equal to the rate of return on real capital, less the cost of financial intermediation that allows deposit-holders to hold their wealth in the more convenient form of deposits rather than real capital"[3] and thus, the cost of holding money being equal to the social cost of creating it, money would be neutral with respect to economic growth. Johnson concludes that "neutrality would be assured by assuming that monetary arrangements guarantee holders of money a rate of return on their real balances equal to the rate of return available on real investment."[4]

However, in his later writings[5] Johnson recognizes his error of having adopted the Gurley-Shaw distinction between inside and outside money with respect to their wealth effects and correctly asserts that the two types of money are "exactly equivalent in their effects on

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[1] Ibid., p. 117.

[2] Ibid.

[3] Ibid., p. 118.

[4] Ibid.

[5] For example, H. G. Johnson, "Inside Money, Outside Money, Income, Wealth, and Welfare in Monetary Theory," Journal of Money, Credit and Banking, I (February, 1969), pp. 30-45.

society's wealth,"[1] as Pesek and Saving have demonstrated.[2] Johnson revises his conclusion made earlier and says that "the implications of a more or less rapid rate of inflation implemented by the monetary authorities for the level of consumption of real goods and services per capita on the steady-state growth path ... are ambiguous,"[3] because the influence of the rate of inflation is a compound of two influences on real income working in opposite directions. The increase in the rate of inflation, on the one hand, lowers the utility yield on money balances and thereby lowers the proportion of national income available for capital accumulation. But, on the other hand, the lowering of the rate of return on real balances caused by an increase in the rate of inflation lowers the ratio of desired real balances to income and hence lowers the proportion of any given amount of savings that will go to the accumulation of real

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[1] Ibid., p. 34.

[2] Boris Pesek and Thomas Saving, Money, Wealth, and Economic Theory. (New York: Macmillan, 1967).

Pesek and Saving are right in contending that both inside and outside money add to the society's wealth and therefore have a wealth effect on consumption and saving. But they are wrong in saying that when the interest equal to the market rate of interest is paid on inside money the inside money loses its moneyness, becoming a debt, and hence do not add to the society's wealth. To say that the value of money becomes zero in this case is the same as to say that the value of water is zero because its market price is zero. The correct way to look at it is to understand that money becomes a free good when the net opportunity cost of holding it becomes zero which would be the case when the market rate of interest is applied to the demand deposits.

For detailed discussion on his point, see: Milton Friedman and Anna Schwartz, "The Definition of Money: Net Wealth and Neutrality as Criteria," Journal of Money, Credit and Banking, I (February 1969), pp. 1-14; H. G. Johnson, "Inside Money, Outside Money, Income, Wealth, and Welfare in Monetary Theory;" and Alvin L. Marty, "Inside Money, Outside Money, and the Wealth Effect," Journal of Money, Credit and Banking, I (February 1969), pp. 101-111.

[3] Johnson, Ibid., p. 40.

balances as contrasted with the accumulation of real capital. Thus an increase in the rate of monetary expansion can either raise or lower the equilibrium capital intensity depending upon which of these two forces dominates.

A similar, but not the same, kind of indeterminacy has been demonstrated by Levhari and Patinkin. In their consumer's good approach, Levhari and Patinkin conclude that the effect of changes in the rate of monetary expansion on the equilibrium capital intensity is indeterminate. This indeterminacy is due to the fact that "on the one hand the increased money rate of interest[1] increases the imputed disposable income from money balances; but on the other hand, the increase in the rate of inflation means a decrease in the rate of return from money balances ..."[2] Nevertheless, they argue that a sufficiently large price decline must necessarily cause a decrease in the equilibrium capital intensity. This must be true for the following reasons: (1) if the production function is well-behaved, then for each value of π there exists a unique nonzero steady-state value of k (as is represented by Curve EE in the diagram in footnote 1, p. 23); (2) under perfect competition the real rate of interest r equals the marginal productivity of capital y'(k) which is a decreasing function of k (as represented by the Curve QQ in the footnote diagram); and (3) in any equilibrium situation the yield on money, $-\pi$, must not be greater than the yield on physical capital, r, for otherwise liquidity

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[1]The increased money rate of interest is caused by the increase in the rate of inflation.

[2]Levhari and Patinkin, op. cit., p. 725. For a mathematical proof, see Appendix III(a).

considerations would cause everyone to shift out of the latter into the former, Levhari and Patinkin conclude that "even though the effect of a change in π on the steady-state value of k is indeterminate in the small, ... a sufficiently large decrease in π must decrease k."[1] However, as far as the effect of a change in π on the steady-state value of m is concerned, the effect is indeterminate. In their producer's good approach, the signs of comparative-dynamic derivatives, $dk/d\pi$ and $dm/d\pi$, are also found to be indeterminate, i.e., the effects of a change in the rate of monetary expansion on the equilibrium values of k and m are indeterminate.[2]

Miguel Sidrauski uses the individual's inter-temporal utility maximization approach (maximizing the total utility over the entire economic horizon, discounted at the subjective rate of time preference) and arrives at the conclusion that the long-run equilibrium capital intensity is independent of the rate of monetary expansion although in the short run an increase in the rate of monetary expansion reduces the rate of capital accumulation.[3] Sidrauski's conclusion that the

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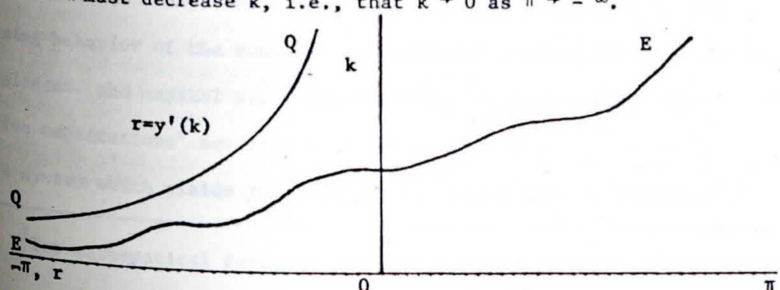
[1] Ibid., p. 727. The diagram shows that a sufficiently large decrease in π must decrease k, i.e., that $k \to 0$ as $\pi \to -\infty$.



[2] For mathematical proofs, see Appendix III(b).

[3] Miguel Sidrauski, "Rational Choice and Patterns of Growth," Journal of Political Economy, 77, No.4, Part II (July/August, 1969). For mathematical proof, see Appendix IV.

equilibrium capital intensity is independent of the rate of monetary expansion in the long run is due to the assumption that the subjective rate of time preference is constant and independent of the stock of wealth and its composition. If the rate of discount depends in some way on the value of m, the equilibrium capital intensity will no longer be independent of the rate of monetary expansion.

Sidrauski postulates a utility functional of the form

$$(1.13)\quad W = \int_0^{\infty} [U(c_t, m_t)] e^{-\delta t} dt$$

where W = total welfare, c = per capita consumption, m = per capita real balances and δ = constant subjective rate of time preference. An individual is assumed to maximize his total welfare W over an infinite time horizon subject to the stock and flow constraints:

$$(1.14)\quad a_t = k_t + m_t$$

$$(1.15)\quad \dot{a}_t = y(k_t) + v_t - (\pi_t + n)m_t - (u + n)k_t - c_t$$

where a = per capita nonhuman wealth, v = real value of net government transfers (outside money) and u = rate of depreciation. From this utility-maximizing behavior of the consumer the demand functions for consumption, cash balances, and capital stock are derived. By introducing Cagan's "adaptive expectations" equation into the model Sidrauski derives a dynamic system which yields the steady-state conditions as follows:[1]

[1]For mathematical derivations, see Appendix IV.

(1.16) $c^* = y(k^*) - (u + n)k^*$

(1.17) $\pi^* = \mu - n$

where c^*, k^* and π^* represent the equilibrium values of c, k and π respectively. From (1.16) we find the equilibrium capital intensity

(1.18) $k^* = \frac{1}{u + n}(y(k^*) - c^*)$

which is independent of the rate of monetary expansion μ or the rate of price change π.

Jerome Stein in his "Keynes-Wicksell" model[1] postulates an independent investment function and assumes that prices change only when there is an excess demand, positive or negative, in the commodity market, i.e., only when planned investment differs from planned saving.[2] Therefore, Stein model is a kind of disequilibrium model. This creates a sharp

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[1]Jerome L. Stein, "'Neoclassical' and 'Keynes-Wicksell' Monetary Growth Model," Journal of Money, Credit and Banking, I (May, 1969), pp. 153-71.

[2]Firms are assumed to desire a capital-labor ratio such that the marginal product of capital (r) is equal to the real rate of interest which is a difference between the nominal rate of interest and the expected rate of price change, $i-\pi^*$. It is assumed that the desired rate of growth of the capital-labor ratio is postively related to the difference between the marginal product of capital and the real rate of interest, $r + \pi^* - i$. Of course, the desired rate of investment will not be such that the marginal product of capital is immediately equated to the real rate of interest. Various lags exist in the investment process. When these two rates become equal, that is, when $r = i - \pi^*$, then the capital-labor ratio remains constant, i.e., the desired rate of investment per unit of capital is equal to n. Thus the desired rate of investment per unit of capital can be expressed as

$$I/K = n + r + \pi^* - i$$

where π^* represents the expected rate of price change.

contrast between Stein model and the models mentioned so far which are all equilibrium models. Another distinguishing feature of Stein model is that it includes bonds market in the model and it also includes inside money as well as outside money. However, inside money is not considered a form of wealth.

In this kind of disequilibrium model, the capital formation is no longer determined solely by the saving hypothesis. Stein assumes that when there is a price inflation the actual growth of capital stock is a linear combination of planned saving and planned investment. The dynamic system in Stein model is represented by a pair of differential equations:[1]

(1.19) $\dot{x}/x = n - \dot{K}/K.$

(1.20) $\dot{\hat{m}}/\hat{m} = \mu - \pi - \dot{K}/K$

where $x = L/K$ and $\hat{m} = M/(PK)$. From (1.19) and (1.20) the steady-state values of x and $\hat{m}$ can be obtained. However, Stein fails to obtain an unambiguous conclusion regarding the comparative-dynamic question and simply states that the steady-state values of capital intensity and per capital real balances "can either fall, rise or remain constant as a result of a rise in the rate of monetary expansion."[2]

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[1]See Appendix V.

[2]Stein, op. cit., p. 167. For mathematical proof, see Appendix V. However, in his earlier model Stein maintains that an increase in the rate of monetary expansion raises the equilibrium capital intensity. See, J. L. Stein, "Money and Capacity Growth," Journal of Political Economy, 74 (October, 1966), pp. 451-65.

$$(3.8)\quad nk = sy(k,m,b) - (1 - s)(m + b)n$$

$$(3.9)\quad \mu - \pi - n = 0$$

$$(3.10)\quad \gamma - \pi - n = 0.$$

The steady-state conditions imply that (a) both capital and effective labor grow at the same rate, i.e., the warranted rate is equal to the natural rate in the Harrodian terminology; (b) the natural rate of growth is equal to the rate of growth of the nominal stock of money minus the rate of price change; and (c) the natural rate of growth is also equal to the rate of growth of the nominal stock of bonds minus the rate of price change. Hence, in the steady state the rate of growth of the nominal stock of money is equal to the rate of growth of the nominal stock of bonds, i.e., $\mu = \gamma$.

From the steady-state conditions (3.8) - (3.10) we can also derive the equilibrium (steady-state) values of k, m and b. To find these values, however, we need further informations concerning the demands for real balances and real bonds. The demands for real balances and real bonds as factors of production are determined by the marginal productivity principle according to which the marginal advantages from each factor must be equal. Since the marginal advantages obtained from holding a unit of real balances consists not only of its marginal product, y_m, but also of the marginal capital gains, $-\pi$, and similarly for bonds denominated in monetary units, the relevant marginal conditions will be

$$(3.11)\quad y_k(k,m,b) = y_m(k,m,b) - \pi$$

$$(3.12)\quad y_k(k,m,b) = y_b(k,m,b) - \pi$$

which implies that the marginal product of real balances must equal the marginal product of real bonds, i.e., that $y_m = y_b$. Therefore, the demand for real balances can be expressed as a function of k, b and π and the demand for real bonds can be expressed as a function of k, m and π, i.e.

(3.13) $m = m(k,b,\pi)$

(3.14) $b = b(k,m,\pi)$.

Now, given the values of parameters, μ, γ, n and s, the equilibrium values of k, m and b can be determined. For any given value of π, the equilibrium values of k, m and b are:

(3.15) $k^* = sy(k^*,m^*,b^*)/n - (1 - s)(m^* + b^*)$

(3.16) $m^* = m(k^*,b^*,\pi)$

(3.17) $b^* = b(k^*,m^*,\pi)$.

The significance of the equation system (3.15) - (3.17) is that the equilibrium capital intensity in the monetary model can be higher than the one in the Solow-type non-monetary model, $sy(k)/n$, since the equilibrium capital intensity in our model is affected not only by the propensity to save but also by the rate of monetary expansion. From (3.15) it is obvious that, if the sum of two effects, k_m and k_b, is positive, the equilibrium capital intensity in the monetary model will be higher.[1] This means that

[1]Equation (3.15) may be rewritten as $k^* = k(m^*,b^*)$.

the equilibrium capital intensity in the monetary model will be higher if the contributing effect of the introduction of monetary assets into the monetary model is stronger than its negative effect on capital intensity. The contributing effect may be referred to as "output" effect, since it increases capital intensity through its indirect effect on output. The negative effect may be referred to as "leakage" effect, since it decreases capital intensity by diverting some part of saving from real capital formation to accumulation of monetary assets. Therefore, if the "output" effect is stronger than the "leakage" effect, the total effect will be positive; otherwise it will be negative. It can be shown that the values of k_m and k_b are identical and, therefore, that the total effect will be positive if and only if k_m is positive.[1] It can also be shown that k_m is positive if $y_m > n(1 - s)/s$. Therefore, we conclude that the equilibrium capital intensity will be higher (lower) in the monetary model if y_m is greater (smaller) than $n(1 - s)/s$.[2]

In a model such as this in which the saving ratio is assumed to be constant, the only policy parameter available is the rate of monetary expansion μ or equivalently the rate of price change π. The variations in the rate of monetary expansion affect not only the equilibrium capital intensity but also the equilibrium values of other real variables, m^* and b^*, as well. Therefore, one of the central questions in the monetary growth theory is to determine the direction of influence of the variations in the rate of monetary expansion on the equilibrium values of real variables, i.e., to determine the signs of the comparative-dynamic derivatives,

[1]For the proof, see Appendix VI.

[2]It will be shown later in this chapter that under normal conditions y_m is smaller than $n(1 - s)/s$ and hence the equilibrium capital intensity is lower in the monetary model.

$dk/d\mu$, $dm/d\mu$ and $db/d\mu$, which are equivalent to $dk/d\pi$, $dm/d\pi$ and $db/d\pi$ respectively.[1] A change in the rate of monetary expansion will alter the time profiles of real variables via (3.5) - (3.7) and hence their equilibrium values as well.

To examine the comparative-dynamic aspects of the growth equilibrium represented by the steady-state conditions (3.8) - (3.10), we differentiate (3.8), (3.11) and (3.12) with respect to π to obtain a system of simultaneous equations[2] involving the variables, $dk/d\pi$, $dm/d\pi$ and $db/d\pi$. $dk/d\pi$, $dm/d\pi$ and $db/d\pi$ are comparative-dynamic derivatives in the sense that they represent the rates of change of k, m and b with respect to the infinitesimal change in the rate of monetary expansion and hence in the rate of price change. By solving these simultaneous equations for $dk/d\pi$, $dm/d\pi$ and $db/d\pi$, we obtain

(3.18) $dk/d\pi = [(1 - s)n - sy_m](2y_{mb} - y_{bb} - y_{mm})/\Delta$

(3.19) $dm/d\pi = \{(n - sy_k)(y_{bb} - y_{mb}) + [(1 - s)n - sy_m](y_{mk} - y_{bk})\}/\Delta$

(3.20) $db/d\pi = \{(n - sy_k)(y_{mm} - y_{bm}) + [(1 - s)n - sy_m](y_{bk} - y_{mk})\}/\Delta$

where Δ is the determinant of the coefficient matrix.[3] Hence, the signs of $dk/d\pi$, $dm/d\pi$ and $db/d\pi$ cannot be determined on a priori ground.

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[1]In the steady state $\mu - \pi = n$ and therefore $d\mu - d\pi = 0$, since n is constant. Hence, in the steady state, $d\mu = d\pi$.

[2]The simultaneous equations are:

$$(n - sy_k)dk/d\pi + [(1 - s)n - sy_m]dm/d\pi + [(1 - s)n - sy_b]db/d\pi = 0$$

$$(y_{kk} - y_{mk})dk/d\pi + (y_{km} - y_{mm})dm/d\pi + (y_{kb} - y_{mb})db/d\pi = - 1$$

$$(y_{kk} - y_{bk})dk/d\pi + (y_{km} - y_{bm})dm/d\pi + (y_{kb} - y_{bb})db/d\pi = - 1$$

[3]For a formal derivation, see Appendix VI.

However, we can safely assume that in a growth equilibrium the economy has already achieved the portfolio balance with respect to the long-run stocks of capital, real balances and real bonds, i.e., that none of these assets is over-accumulated or under-accumulated in relation to each other. This implies that, when a small amount of any one factor is added to the economy while the amount of other factors remaining constant, that extra addition of one factor will increase the marginal products of the remaining factors in the same proportion.[1] This is a reasonable assumption to make in the steady state. The fact that the marginal products of the remaining factors increase in the same proportion as a result of a small amount increase in any one factor implies that the second-order cross partial derivatives in the production function are all equal,[2] i.e., $y_{km} = y_{mk} = y_{bk} = y_{kb} = y_{mb} = y_{bm}$.

Now we can determine the signs of the comparative-dynamic derivatives. First of all, we note that $(n - sy_k)$ is positive.[3] The common

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[1]A similar assumption is made by J. R. Hicks when he discusses technical progress. Hicks-neutral technological progress means that the marginal products of labor and capital increase by the same amount as a result of technical progress. See J. R. Hicks, The Theory of Wages (London: Macmillan, 1932), pp. 121-27.

[2]The marginal product of the factor whose amount has been increased will, of course, decrease.

[3]The Golden Rule requires $n = y_k$, which implies $(n - sy_k) > 0$ in the optimum state, since $0 < s < 1$. Even in the non-optimum state, the natural rate of growth will normally exceed the product of saving ratio and marginal productivity of capital. The extreme case in which this is not true will be discussed later in this chapter.

denominator Δ[1] will be positive $(1 - s)n - sy_m > 0$, i.e., if $y_m < n(1 - s)/s$. The numerator of $dk/d\pi$ is positive, if $y_m < n\frac{(1 - s)}{s}$. The numerators of $dm/d\pi$ and $db/d\pi$ are all negative.[2] Therefore, if y_m is smaller than $n(1 - s)/s$, $dk/d\pi$ will be positive and $dm/d\pi$ and $db/d\pi$ will be negative. The significance of this conclusion is that, as long as y_m is smaller than $n(1 - s)/s$, an increase in the rate of monetary expansion will result in a rise in the equilibrium capital intensity and a fall in the equilibrium values of per capita real balances and real bonds. However, our conclusion heavily depends on the condition that y_m must be smaller than $n(1 - s)/s$. Fortunately, it can be shown that under normal conditions y_m is in fact smaller than $n(1 - s)/s$. To show that it is so, the possible combinations of different values of the parameters involved are shown in Table 3-1.

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[1]The common denominator Δ is $(n - sy_k)[(y_{km} - y_{mm})(y_{kb} - y_{bb}) - (y_{km} - y_{bm})(y_{kb} - y_{mb})] + [(1 - s)n - sy_m][(y_{kb} - y_{mb})(y_{kk} - y_{bk}) + (y_{kk} - y_{mk})(y_{km} - y_{bm}) - (y_{km} - y_{mm})(y_{kk} - y_{bk}) - (y_{kk} - y_{mk})(y_{kb} - y_{bb})]$ which is, due to the equality of all the second-order cross partial derivatives, equal to $(n - sy_k)(y_{km} - y_{mm})(y_{kb} - y_{bb}) - [(1 - s)n - sy_m][(y_{km} - y_{mm})(y_{kk} - y_{bk}) + (y_{kk} - y_{mk})(y_{kb} - y_{bb})]$ which will be positive, if $(1 - s)n - sy_m > 0$.

[2]The numerator of $dk/d\pi$ is $[(1 - s)n - sy_m](2y_{mb} - y_{bb} - y_{mm})$ which is positive if $(1 - s)n - sy_m > 0$, since $y_{mb} > 0$, $y_{bb} < 0$ and $y_{mm} < 0$. The numerator of $dm/d\pi$ is $(n - sy_k)(y_{bb} - y_{mb}) + [(1 - s)n - sy_m](y_{mk} - y_{bk})$ which is negative if $n - sy_k > 0$, since $y_{bb} < 0$, $y_{mb} > 0$ and $y_{mk} = y_{bk}$. The numerator of $db/d\pi$ is $(n - sy_k)\cdot(y_{mm} - y_{bm}) + [(1 - s)n - sy_m](y_{bk} - y_{mk})$ which is negative for the same reason.

TABLE 3-1

DIFFERENT VALUES OF n(1 - s)/s

s	1-s	$\frac{1-s}{s}$ \ n	.025 (.005)*	.03 (.01)	.032 (.012)	.035 (.015)	.038 (.018)	.04 (.02)	.042 (.022)	.045 (.025)	.048 (.028)	.05 (.03)
.05	.95	19.0	.48	.57	.61	.67	.72	.76	.80	.86	.91	.95
.08	.92	11.5	.29	.35	.37	.40	.44	.46	.48	.52	.55	.58
.10	.90	9.0	.23	.27	.29	.32	.34	.36	.38	.41	.43	.45
.12	.88	7.3	.18	.22	.23	.26	.28	.29	.31	.33	.35	.37
.15	.85	5.7	.14	.17	.18	.20	.22	.23	.24	.26	.27	.29
.18	.82	4.6	.12	.14	.15	.16	.18	.18	.19	.21	.22	.23
.20	.80	4.0	.10	.12	.13	.14	.15	.16	.17	.18	.19	.20
.25	.75	3.0	.08	.09	.10	.11	.11	.12	.13	.14	.14	.15

*The figures in the parentheses represent the rate of population growth. The rate of technological progress is the difference between the natural rate and the rate of population growth.

Since the saving ratio will normally range from .08 to .12 and the natural rate of growth (the sum of the rate of population growth and the rate of technological progress) from .03 to .05 in an advanced capitalistic economy, the value of n(1 - s)/s will normally range from .22 to .58 in an advanced capitalistic economy, as shown in Table 3-1. If we consider the marginal productivity of real balances, y_m, as ranging from .05 to .15 in an advanced capitalistic economy, the condition y_m will be satisfied for all reasonable values of n, s and y_m. For example, suppose that the saving ratio is .10, that the rate of population growth is .015 and that the rate of technological progress (productivity growth) is .025. Then the value of n(1 - s)/s will be .36 which may be considered as a typical value in American economy. (This figure is circled in Table 3-1). There is no chance that y_m might exceed .36. However, if s is extremely high, say .20 and n is extremely low, say .025, although this is very unlikely, then it might be possible for $n(1 - s)/s$ to be smaller than y_m which will be .10 in this case. The fact is that, even if this extreme case does occur, in which $y_m > n(1 - s)/s$, the signs of comparative derivatives will still remain the same, because the extremely large value of s and extremely small value of n will make $(n - sy_k)$ negative.[1] When $n - sy_k < 0$ and $y_m > n(1 - s)/s$, the common denominator becomes negative and the numerators become negative for $dk/d\pi$ and positive for $dm/d\pi$ and $db/d\pi$. Therefore, the signs of these derivatives will remain the same. Our final conclusion is that for a steadily growing advanced capitalistic economy an increase in the rate of monetary expansion will result in a rise in the equilibrium capital intensity and a fall in the equilibrium values of per capita real balances and real bonds, and conversely for a decrease in the rate of monetary expansion. We interpret this as an evidence that the stabilizing Pigou effect eventually wins out the destabilizing Wicksell effect in case of deflationary spiral.[2]

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[1]Notice that a high value of y_m also implies a high value of y_k for a given value of π, since $y_k = y_m - \pi$. This means that an extremely low n and an extremely high s, for y_m large enough to make itself greater than $n(1 - s)/s$, will certainly make $(n - sy_k)$ negative.

[2]An accelerated price decline has two effects, at war with each other. On the one hand, by augmenting the real value of existing money balances, it helps to restore portfolio balance and, by increasing total real wealth, retards the flow of saving into capital formation. [Pigou effect]. On the other hand, it means a more attractive yield on money and, therefore, encourages a further shift in portfolio demand in the same direction as the original shock. [Wicksell effect]. Therefore, if the Pigou effect is stronger than the Wicksell effect, the economy will, after an accelerated price decline, settle down with a lower capital intensity. See Tobin, "Money and Economic Growth," pp. 682-3.

It is also interesting to note that, if $n - sy_k > 0$ and $y_m < n(1 - s)/s$, the equilibrium capital intensity will be lower in the monetary model. Furthermore, even if $n - sy_k < 0$, the equilibrium capital intensity in the monetary model will be lower, because when $n - sy_k < 0$ it also follows that $y_m > n(1 - s)/s$.[1] It has been shown that the positive (negative) value of k_m implies that the equilibrium capital intensity will be higher (lower) in the monetary model. Since $k_m = [sy_m - n(1 - s)]/(n - sy_k)$,[2] the equilibrium capital intensity will be lower in the monetary model, if $n - sy_k > 0$ and $y_m < n(1 - s)/s$, or if $n - sy_k < 0$ and $y_m > n(1 - s)/s$. Thus we also conclude that the equilibrium capital intensity is lower in the monetary model. This does not, of course, mean that the equilibrium capital intensity in a monetary economy is lower than what would have been in a barter economy.

In this chapter it has been shown, by using the equilibrium approach, that the equilibrium capital intensity will be lower in the monetary model and that an increase in the rate of monetary expansion will result in a rise in the equilibrium capital intensity and a fall in the equilibrium values of per capita real balances and per capita real bonds.

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[1]See footnote 2, p. 50.

[2]See Appendix VI.

CHAPTER IV

THE COMPARATIVE-DYNAMIC ASPECTS OF GROWTH

EQUILIBRIUM II: A DISEQUILIBRIUM MODEL

In Chapter III an equilibrium approach was used to examine the comparative-dynamic aspects of the model economy introduced in Chapter II. It was shown that in a steadily growing advanced capitalistic economy an increase in the rate of monetary expansion results in an increase in the equilibrium capital intensity and a decrease in the equilibrium values of per capita real balances and real bonds. In this chapter a different approach will be used to examine the question of comparative dynamics, which may be called a "disequilibrium approach."[1]

We no longer assume that the capital formation is determined solely by savings hypothesis and monetary expansion. Instead, we introduce an independent investment function which determines, together with the saving function, the actual growth of capital stock, given the state of institutional market arrangements which determines how realized saving (or investment) deviates from planned saving (or investment) in times of commodity market disequilibrium. We no longer assume that all markets are in equilibrium at all times but, instead, assume that the excess demand, positive or negative, normally exists in the commodity and money markets, although we continue to assume that the labor market is always in equilibrium.[2]

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[1]A disequilibrium approach has been used by a number of authors. For example, J. L. Stein, "'Neoclassical' and 'Keynes-Wicksell' Monetary Growth Models," op. cit. and H. Rose, "On the Non-Linear Theory of the Employment Cycle," Review of Economic Studies, XXXIV (April, 1967), pp. 153-74.

[2]The case in which the possibility of unemployment is existent is considered by H. Rose [illegible]

It is assumed that the speculative activities are so strong and the speed of response is so rapid in the bonds market that the excess demand in this market is instantaneously eliminated and the equilibrium is immediately restored. This implies, according to the Walras' Law, that the excess demand (supply) in commodity market must be equal to the excess supply (demand) in money market for any given period of time. A disequilibrium in the commodity and money markets creates a movement of general price level which causes the actual investment to deviate from the planned one. Therefore, in a disequilibrium model the path of capital accumulation is generally affected by the market phenomena.

Although our methodology has changed, our model economy has not. We continue to assume that there are four markets and that money and bonds are factors of production while they are additions to the community's wealth, whether they are inside or outside type. Therefore, the model to be presented in this chapter differs from Stein model in several respects, although the fundamental approach used is the same. First, the production function in our model is of the form $Y = Y(K,L,M/P,B/P)$ where both real money balances and real bonds are treated as factors of production whereas the production function in Stein model is a conventional two-factor production function where capital and labor are the only factors of production. Second, all the variables in our model are expressed in per-capita terms (or in terms of per-unit-of-effective-labor) whereas the variables in Stein model are expressed in terms of per-unit-of-capital.

In this model we do not assume that saving is a constant proportion of real disposable income. Instead, it is assumed that planned saving, S, is a function of the present value of human income, the stock of nonhuman wealth, and the real rates of return on capital, money balances and bonds. In a disequilibrium approach it seems to be more desirable to allow for the effects on planned saving of variations in the income and wealth and of the rates of return on various forms of assets explicitly rather than to assume a constant saving ratio. The effects of the rates of return on various forms of assets on the level of planned saving are, however, assumed to be negligible, although they may play important roles in determining what portion of saving is devoted to which of the various forms of wealth. We use the current output as a proxy for the present value of human income so that planned saving becomes a function of current output and the stock of nonhuman wealth. Since the output is a function of K, M/P and B/P, and the stock of nonhuman wealth is the sum of K, M/P and B/P, the planned saving can be expressed, in per capita terms, as

(4.1) $S/L = S(k,m,b)$.

The planned investment, on the other hand, depends on the expected rate of return on capital and the expected rates of return on other forms of wealth or the expected opportunity cost. More specifically, it is an increasing function of the difference between the expected rate of return on capital, y_k, and the expected opportunity cost, $y_m - \pi^*$ or $y_b - \pi^*$, where π^* is the expected rate of price change. Of course, in our disequilibrium model it does not usually follow that $y_k = y_m - \pi^*$, although it is assumed that $y_m = y_b$, i.e., that the equality between the rates of return on money and bonds is maintained. Thus we have

(4.2) $I/L = I(y_k - y_m + \pi^*)$; $I' > 0$.

We assume that prices change only when there is an excess demand in the commodity market. The actual rate of price change, π, is positively related to the excess demand in the commodity market or to the excess supply in the money market so that

(4.3) $\pi = \beta(I/L - S/L)$

where β is some constant representing the speed of market response. The expected rate of price change, π^*, is assumed to be positively related to the current rate of price change although in the steady state the actual and expected rates are equal.

When there is a positive excess demand in the commodity market there must be a positive excess supply (negative excess demand) in the money market since the labor and bonds markets are assumed to be always in equilibrium.[1] The demand for real money balances as a factor of production depends on its expected real rate of return and the expected real rates of return on other forms of wealth or the expected opportunity cost. It is an increasing function of the difference between the expected rate of return on money, $y_m - \pi^*$, and the expected opportunity cost, y_k, i.e.

(4.4) $m^d = m^d(y_m - \pi^* - y_k)$; $m^{d\prime} > 0$.

Assuming that the flow excess supply of real balances is proportional to the stock excess supply, $m - m^d$, we have

[1]According to the Walras' Law, the sum of excess demands in all markets must be zero.

$$(4.5)\quad \pi/\beta = h[m - m^d(y_m - \pi^* - y_k)]$$

where h is a factor of proportionality. Equation (4.5) implies that the flow excess demand in commodity market, π/β, is equal to the flow excess supply in money market, $h(m - m^d)$. Therefore, the actual rate of price change, π, is determined by (4.3) and (4.5). When the monetary authorities decide the supply of money, the amount of excess demand, or supply, of money is determined depending upon what the demand conditions are. If there exists a non-zero excess supply of money, there will also be a non-zero excess demand for commodities. Then the prices will rise.

When the prices are rising, the investors' and savers' plans will not be fully realized. In other words, in the periods of excess aggregate demand in the Keynesian sense, the actual investment will not be equal to the planned investment or planned saving. We assume that when prices are rising the actual growth of capital stock is a linear combination of planned investment and planned saving, but, when prices are remaining constant or falling, the actual investment is equal to planned saving. Thus we have

$$(4.6)\quad \dot{K}/L = \alpha(I/L) + (1 - \alpha)(S/L);\; 0 < \alpha < 1 \text{ when } \pi > 0$$
$$\alpha = 0 \text{ when } \pi \leq 0$$

where α reflects the current state of institutional market arrangements which determines how output is distributed between savers and investors in the periods of excess aggregate demand. Equation (4.6) implies that in the periods of stable or falling prices the saving function determines the actual growth of capital but in the periods of rising prices the actual growth of capital lies somewhere between planned saving and planned investment.

The demand for bonds in real terms (real bonds) as a factor of production is also assumed to be, like the demand for any other factor, an increasing function of the difference between its real rate of return and opportunity cost so that

(4.7) $b = b^d(y_b - \pi^* - y_k)$

since it is assumed that the bonds market equilibrium is instantaneously achieved.

Then what is the time path of capital intensity? And what are the time paths of per capita real balances and per capita real bonds? To answer these questions we derive (4.8) from (4.3) and (4.6):[1]

(4.8) $\dot{K}/K = (1/k)(\alpha\pi/\beta + S/L)$

from which (4.9) is derived:

(4.9) $\dot{k} = \alpha\pi/\beta + S(k,m,b) - nk.$

From $m = M/PL$ and $b = B/PL$, we obtain

(4.10) $\dot{m} = (\mu - \pi - n)m$

(4.11) $\dot{b} = (\gamma - \pi - n)b.$

Differential equations (4.9) - (4.11) constitute a dynamic system of our model economy which describes the time profiles of real variables, k, m, and b.

[1]For a mathematical proof, see Appendix VII.

From the dynamic system represented by (4.9) - (4.11) we obtain the steady-state conditions by setting $\dot{k} = 0$, $\dot{m} = 0$ and $\dot{b} = 0$, that is,

(4.12) $nk = \alpha\pi/\beta + S(k,m,b)$

(4.13) $\mu - \pi - n = 0$

(4.14) $\gamma - \pi - n = 0$

which imply that in the steady state all real variables grow at the same constant rate n. Therefore, the equilibrium capital intensity k^* is

(4.15) $k^* = (\alpha\pi)/(\beta n) + (1/n)S(k,m,b)$.

Let us now examine the comparative-dynamic properties of the growth equilibrium characterized by (4.12) - (4.14). To do so we first note that the actual rate of price change π can be expressed as a function of k, m and b only, i.e., that $\pi = \pi(k,m,b)$ from (4.1) - (4.3). Since π is a function of excess demand in the commodity market (or equivalently excess supply in the money market) and since both planned saving and planned investment are a function of k, m, and b, the rate of price change π can be expressed as a function of k, m and b only. Thus (4.13) can be written as

(4.16) $\mu - \pi(k,m,b) - n = 0$.

Since $\mu = \gamma$ in the steady state, either (4.13) or (4.14) is redundant in our dynamic system. Therefore, we have only two equations, (4.12) and (4.16), but three unknowns, k, m and b. One more equation is needed to make the system soluble. We can add the equation stating that the real rates of return on money and bonds are equal, i.e.

(4.17) $y_m(k,m,b) = y_b(k,m,b)$.

Now a system of simultaneous equations[1] can be obtained by differentiating (4.12), (4.16) and (4.17) with respect to μ, which can be solved for $dk/d\mu$, $dm/d\mu$ and $db/d\mu$. The solutions are:

(4.18) $dk/d\mu = \{[(\alpha/\beta)\pi_m + S_m](y_{mb} - y_{bb}) - [(\alpha/\beta)\pi_b + S_b](y_{mm} - y_{bm})\}/\Delta$

(4.19) $dm/d\mu = \{[n - (\alpha/\beta)\pi_k - S_k](y_{mb} - y_{bb}) + [(\alpha/\beta)\pi_b + S_b](y_{mk} - y_{bk})\}/\Delta$

(4.20) $db/d\mu = \{[n - (\alpha/\beta)\pi_k - S_k](y_{bm} - y_{mm}) + [(\alpha/\beta)\pi_m + S_m](y_{bk} - y_{mk})\}/\Delta$

where Δ is the determinant of the coefficient matrix. The signs of these derivatives are ambiguous. There is no way of telling whether an increase in the rate of monetary expansion will increase or decrease the equilibrium values of real variables. However, one thing is clear, That is, in a disequilibrium model of this kind, unlike an equilibrium model the signs of these derivatives depend heavily on the numerical values of α and β among other things. In other words, the state of institutional market arrangements which determines how output is distributed between savers and investors in the periods of rising prices, α, and the speed of market response in eliminating excess demands, β, can have a decisive effect in determining the direction and the size of influence of the change in the rate of monetary expansion on the equilibrium values of real variables, k, m and b. This is, of course, due to the fact that our model is a disequilibrium model.

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[1]The simultaneous equations are:

$[n - (\alpha/\beta)\pi_k - S_k]dk/d\mu - [(\alpha/\beta)\pi_m + S_m]dm/d\mu - [(\alpha/\beta)\pi_b + S_b]db/d\mu = 0$

$\pi_k\, dk/d\mu + \pi_m\, dm/d\mu + \pi_b\, db/d\mu = 1$

$(y_{kk} - y_{bk})dk/d\mu + (y_{mm} - y_{bm})dm/d\mu + (y_{mb} - y_{bb})db/d\mu = 0.$

For further details, see Appendix VII.

Our conclusion in this chapter is that the comparative-dynamic aspects of the steady-state equilibrium in a disequilibrium model are ambiguous but it is clear that the effects of a change in the rate of monetary expansion on the equilibrium capital intensity and on the equilibrium values of other real variables are heavily dependent upon the current state of market arrangements and the speed of market response to disequilibrating disturbances. The ambiguity involved here is primarily due to the complicated nature of disequilibrium model, particularly due to the fact that there is no economic mechanism to determine α and β.[1] Indeed, we must agree with Frank Hahn who says; "One has, in the present state of knowledge, great latitude in the construction of disequilibrium models; that is one of the reasons why they are so unattractive and so great a variety of result can be produced."[2]

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[1]Note that the ambiguity involved in this disequilibrium model will remain even if a constant saving ratio is assumed as long as there is no economic mechanism to determine α and β.

[2]Frank Hahn, op. cit., p. 186.

CHAPTER V

THE OPTIMAL RATE OF MONETARY EXPANSION

So far we have been dealing with the comparative-dynamic aspects of growth equilibrium. In this chapter we discuss some important aspects of optimal monetary growth in the equilibrium model developed in Chapter III.

It is defined that a growth path is optimal if it is associated with the greatest total utility either at every instant of time or over the entire economic horizon. Therefore, there are two concepts of optimality in a strict sense. One states that a growth path is optimal if it is associated with the highest constant level of utility per unit of time. The other states that the optimum growth path is the one that maximizes the total utility discounted at an appropriately chosen subjective rate of time preference over the entire economic horizon. In Chapter I we have seen that the first definition of optimality is used by Levhari and Patinkin and the second definition is used by Sidrauski. We have chosen the first definition in this chapter although it is possible to carry out the whole analysis in terms of the second definition.[1]

[1]The Harrod's concept of optimum growth implies that the socially optimal rate of growth is "a maximum rate of growth of output consistent with the full employment of growing labor population and the rising trend of labor productivity." See K. K. Kurihara, The Keynesian Theory of Economic Development (London: Allen and Unwin, 1959), p. 44. The natural rate of growth is considered as an optimal rate in this sense. Since Ramsey's 1928 article, it has been well-known that along the society's welfare-maximizing (optimal) growth path the rate of profit is equal to the natural rate of growth (plus the subjective rate of time preference, if any). Ramsey borrowed this idea from Keynes and developed it. See F. P. Ramsey, "A Mathematical Theory of Saving," The Economic Journal, XXXVIII (December, 1928), pp. 433-4. Hence, on a Ramsey-Keynes optimal growth path, the Harrod's natural rate of growth becomes the society's welfare-maximizing rate, if the society has a zero rate of time preference.

Recently in the literature of monetary growth theory a slightly different concept of optimality has been used largely in the context of optimum stock of money. This concept of optimality states that the optimum stock of money is reached when money is demanded at its satiety level where the marginal product of money is zero,[1] provided that money is costless to produce. Then the optimal rate of growth of money is the rate which keeps the stock of money at this optimum level, i.e., at the satiety level. However, this concept of optimality is deficient because it is not based upon the principle of utility maximization. Therefore, it cannot be considered a valid concept of optimality unless it can be shown that the total utility is maximized at the satiety stock of money. It will be shown at the end of this chapter that the satiety stock is not necessarily the optimum stock and, therefore, the rate of growth of money which keeps the stock of money at the satiety level is not necessarily the optimal rate of growth of money.

Then, on what basis can the satiety stock of money be considered an optimum stock of money? A very illuminating discussion on this question is given by Harry G. Johnson.[2] Starting from the barter conditions represented by the left-hand side of Figure 5-1, the community gradually increases its productivity and real income by introducing commodity money and then replacing commodity money with credit money. The optimum stock of credit money of the community is said to be attained when the satiety level of the stock of money is reached where the marginal utility of money is zero.[3].

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[1]At the satiety level, money becomes a free good.

[2]H. G. Johnson, "Inside Money, Outside Money, .., "JMCB, I (February, 1969).

[3]Money is considered as a consumer's good in Johnson's discussion.

It is, of course, not certain whether this optimum stock represents the true optimum in the sense of maximizing total utility per unit of time or over the entire economic horizon. The argument runs as follows.

Under barter conditions the given stock of material wealth (capital) Ok_0 is allocated entirely to the production of goods and services yielding an income r_0k_0.[1] However, the community finds it extremely difficult to carry out barter transactions. If some form of money (numéraire) could be introduced into the system, its marginal utility would be very high. This fact is expressed by the demand curve for money DD' in Figure 5-1.

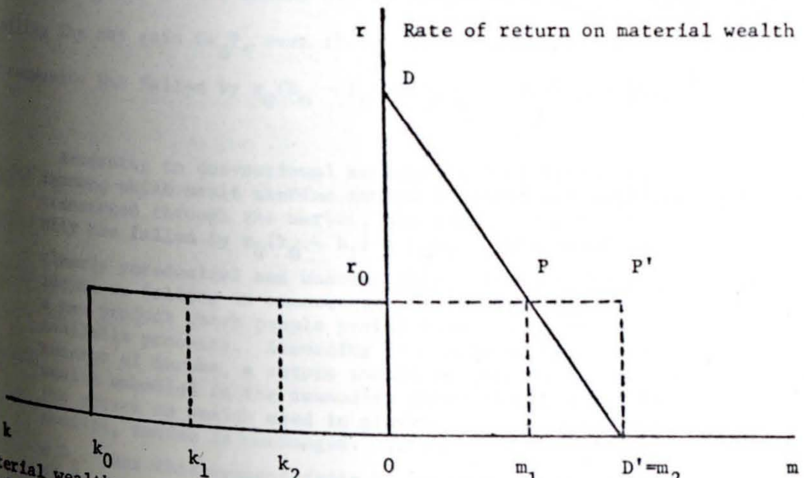

Figure 5-1.--Effect of Introduction of Commodity and Credit Money

[1] I have expressed all the variables in per-capita terms so that the labor input is automatically incorporated into the analysis. Johnson simply ignores the labor input. (See Ibid., p. 32.)

Hence, the community decides to introduce commodity money causing the reallocation of its stock of wealth such that $k_0k_1 = Om_1$ of it is transferred from the stock of productive wealth to the monetary stock. This can be done by using one type of productive capital equipment as a medium of exchange and store of value instead of as an input into the production process. By making a simplifying assumption that "the rate of return on capital is unaffected by variations in the ratio of capital to labor in the production of goods and services of the magnitude entailed by the invention of money,"[1] Johnson argues that as a result of the introduction of commodity money the community's real income has increased from r_0k_0 to $r_0k_0 + Dr_0P$, entailing the net gain Dr_0P, even though it may appear that the income of the community has fallen by $r_0(k_0 - k_1) = r_0m_1$. Johnson argues:[2]

> According to conventional accounting definitions of income, which would confine income to goods and services transacted through the market, the income of the community has fallen by $r_0(k_0 - k_1) = r_0m_1$. This result is clearly paradoxical and unacceptable, since it shows income as falling in consequence of the introduction of a new product which people prefer over the previously available products. According to a more sophisticated concept of income, a return should be imputed to the wealth embodied in the commodity money stock, equal to the return on wealth used in production. On this concept, income is unchanged: it is $r_0k_1 + r_0m_1 = r_0k_0$. But this concept fails to capture the increase in real income and welfare that results from the introduction of money. ... this gain is the triangular consumer's surplus area Dr_0P.

[1] Ibid., p. 32.

[2] Ibid., pp. 33-4. Alternatively, the triangular consumer's surplus area can be considered as the increase in real income resulting from the increased productivity of capital made possible by the introduction of commodity money. This point will be discussed in greater detail later in this chapter.

Now the community realizes that, by replacing the commodity money by credit money (non-interest-bearing), that is, by freeing the material wealth embodied in commodity money so that it can be used in the production of goods and services, it can further increase its real income, if the cost of creating credit money is zero (or very small). After replacing the commodity money entirely with the credit money,[1] the community's real income will have increased from $r_0k_0 + Dr_0P$ to $r_0k_0 + DPm_10$, another extra gain being r_0m_1, since the freed material wealth previously embodied in commodity money is reallocated to the production of goods and services. In Johnson's words, "Income in the strictly conventional accounting sense returns to the barter economy level r_0k_0; income in the more sophisticated accounting sense, imputing an alternative opportunity cost return to holdings of money, is higher than the barter and commodity money level r_0k_0 by r_0m_1; and income in the real income or welfare sense is higher than in the barter economy by DPM_10 and than in the commodity money economy by r_0m_1."[2]

However, this is certainly not the best situation in which the community can be. As long as the marginal utility of money remains positive, the real income of the community can be increased further by inducing the public to increase their holdings of money balances. This can only be done if the holders of money are guaranteed the rate of return on money which is at least equal to the rate of return on material wealth. Now, suppose that the banks are allowed to pay the interest on demand

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[1]It is assumed that the state or the banking system is able to command public confidence without holding any commodity money reserves.

[2]*Ibid.*, pp. 35-6.

deposits and that the competition makes them pay the interest on money such that the rate of interest is equal to the rate of return on material wealth.[1] Then the demand for money will increase up to the point where the marginal utility becomes zero, i.e., equal to the cost of producing money. However, the real rate of return on money will be equal to the rate of return on material wealth r_0. The total interest payment to the holders of money equal to $r_0 m_2$ comes from the production of goods and services and is equal to the output produced by $(k_0 - k_2)$ of material wealth, $r_0(k_0 - k_2)$. Paradoxically, it appears that $m_2 = k_0 - k_2$ of wealth has disappeared and the community is no wealthier than it was under barter conditions. "It also appears, by the strict income accounting convention, that the community's income is the same as under barter conditions; and the more sophisticated income accounting convention implies that the community's income has been reduced to the barter economy level along with its wealth, because the marginal utility value to be imputed to the quality of 'moneyness' is now reduced to zero."[2] However, in reality the community's real income exceeds its barter level $r_0 k_0$ by the amount DOD' which is the satiation level of consumers' surplus on its holdings of money balances. This satiety stock of money is considered the optimum stock since there is no room for further increase of total utility in this situation as far as the utility of money balances is concerned. As will be seen later, this is not true

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[1]It is assumed that it costs nothing to produce money and that the purchasing power of money is not altered by the whole process.

[2]*Ibid.*, pp. 36-7.

because it may be that the total utility derived from both the consumption of goods and services and money balances is greater at the non-satiety level of the stock of money than at the satiety level.

Johnson's argument presented above treats money as a consumer's good and hence the whole discussion is carried out in terms of consumers' surplus. It is possible to treat money as a producer's good and carry out the analysis in terms of productivity gains. For this purpose it is essential to abandon the Johnson's assumption that "the rate of return on capital is unaffected by variations in the ratio of capital to labor in the production of goods and services of the magnitude entailed by the invention of money,"[1] since we are concerned with the productivity gains themselves resulting from the introduction of money or from the increase in the stock of money. Therefore, Figure 5-1 is no longer relevant for our discussion. The modified version of Figure 5-1 is shown below as Figure 5-2.

We start again with the barter conditions under which the given stock of material wealth (capital) Ok_0 is allocated entirely to the production of goods and services, yielding an income r_0k_0. Under this situation, if some form of money could somehow be introduced, its marginal productivity would prove to be infinitely large as Curve B indicates. Therefore, this situation cannot be considered an equilibrium situation. The community will decide to use some part of its material wealth as commodity money. But how much? An obvious answer is that the community will divert material wealth from the production of goods and

[1] Ibid., p. 32.

services to commodity money until the marginal productivity of money becomes equal to the marginal productivity of material wealth (the rate of return on capital). However, notice that in the meantime the marginal productivity of capital schedule shifts upward from A to A' because of the introduction of commodity money. And the marginal productivity of money schedule shifts downward from B to B' because there is less and less amount of capital for each unit of money balances as more and more of material wealth becomes commodity money.[1] Therefore, an equilibrium is reached when $k_0 - k_1 = 0m_1$ amount of material wealth has been converted into commodity money and where the marginal productivities of both capital

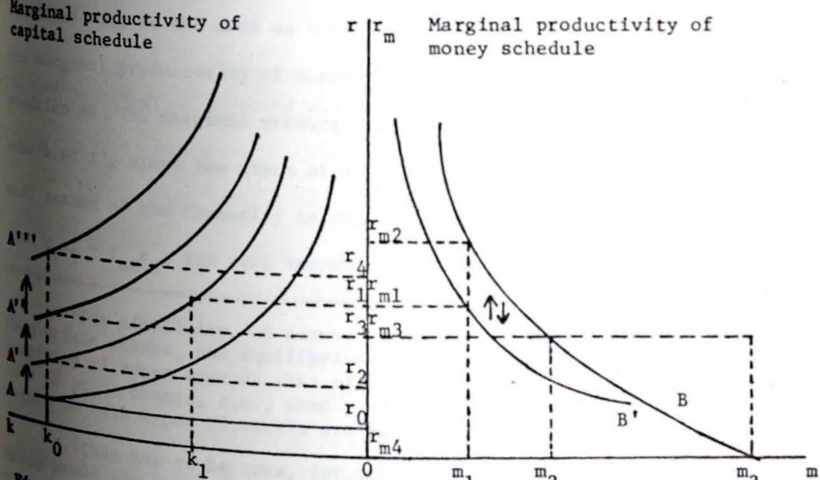

Figure 5-2. Effects of Introduction of Commodity and Credit Money

[1] Again we express all the variables in per-capita terms so that the labor input is automatically incorporated into the analysis.

and money are the same.[1] The real income of the community is $r_1k_1 + r_{m1}m_1$ which is greater than r_0k_0, the real income under barter conditions.[2] The difference between $r_1k_1 + r_{m1}m_1$ and r_0k_0 may be interpreted as being identical with the triangular consumers' surplus area in Figure 5-1. The only difference between the two concepts is that the one is measured in terms of utility and the other is measured in terms of physical output.

Now the community realizes that it could increase its real income if the material wealth embodied in commodity money could be diverted into the production process by replacing it by some form of credit money which is costless to produce (or whose production cost is very small). If the entire amount of commodity money is replaced,[3] the material wealth of the community will be the same as under barter conditions, Ok_0. Therefore, the marginal productivity of money schedule will return to the original position B. The marginal productivity of capital schedule will, however, remain at A', since the stock of money remains unchanged. Hence, the real income of the community is equal to $r_2k_0 + r_{m2}m_1$ which is greater than $r_1k_1 + r_{m1}m_1$, the real income under commodity-money conditions.[4]

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[1]For simplicity, we assume that prices remain stable in this process. When prices change, the equilibrium will be reached when the marginal productivity of money exceeds that of material wealth by an amount equal to the rate of price change, i.e., when $y_k = y_m - \pi$ so that the real rates of return on both types of assets are equal.

[2]This has to be true, for otherwise the introduction of commodity money would not have taken place. The very reason for introducing commodity money was the extremely high marginal productivity of money exceeding the marginal productivity of capital by a great margin as the demand curve for money balances (the marginal productivity of money schedule) shows.

[3]It is assumed that the purchasing power of money is unaffected by this process.

[4]The extra net gain of real income may be compared to the rectangular area r_0m_1 in Figure 5-1.

The fall in real income due to the reduction of marginal productivity of capital r_1 to r_2 will be much smaller than the increase in real income due to the rise in the stock of capital from k_1 to k_0 and to the rise in the marginal productivity of money from r_{m1} to r_{m2}. However, the discrepancy between the marginal productivities of capital and money ($r_{m2} > r_2$) proves that this situation is out of equilibrium. It is to the advantage of the community to create more credit money whose marginal productivity is greater than the marginal productivity of capital. As more and more credit money is added to the existing stock of credit money, the marginal productivity of capital schedule will shift upward further and further, from A' to A". The new equilibrium will be reached at the point where the marginal productivities are equal ($r_3 = r_{m3}$), the community's stock of money now being equal to Om_2 and the real income of the community being equal to $r_3k_0 + r_{m3}m_2$ which is even greater than $r_2k_0 + r_{m2}m_1$, the real income with the stock of money Om_1.

Then, is this equilibrium optimal? No, it is not. Since it is assumed that it costs nothing to create credit money, the community can increase its real income by increasing the stock of money even further so long as its marginal productivity remains positive. As the stock of money is increased due to the creation of more credit money, the marginal productivity of capital schedule will shift upward from A'' to A'''. Once the stock of money has reached the level Om_3 at which the marginal productivity of money is zero or equal to the cost of producing it, there will be no further gain of real income to be made by increasing the stock of money. Therefore, the stock of money Om_3 which may be called the satiety stock may be considered an optimum stock in the sense that there is no more real income to be gained by increasing the stock of money. Then the optimum stock of money in this sense is $0m_3$ and the marginal productivities of capital and money in this situation are r_4 and $r_{m4}(= 0)$ respectively. The real income of the community is equal to r_4k_0 which is even greater than $r_3k_0 + r_{m3}m_2$, real income at the previous equilibrium position. However, this situation is clearly not an equilibrium position since the marginal productivity of capital is much greater than that of money, i.e., $r_4 > r_{m4} = 0$. In other words, the optimum position will not simply be reached unless some measures are undertaken to guarantee the holders of money a rate of return at least equal to the rate of return on capital. The optimum position can be made an equilibrium position either by deflating the general price level continuously at a rate equal to the marginal productivity of capital[1] or by paying the interest on money of the same rate, or by the combination of both. By using the combination of both methods, for example, the optimum equilibrium position is attained where the rate of return on capital is equal to the real rate of return on money balances, $i - \pi$, where i represents the nominal rate of interest on money. In fact, this is the mechanism by which the amount $(r_4 - r_3)k_0$ of real income is diverted from the owners of material wealth to the holders of money balances. Thus, at the satiety stock of money (optimum stock in the present sense) the capital owners' share in real national income is r_3k_0 and the money holders' share is $(r_4 - r_3)k_0$. The total real national income (per capita) is

[1]At this point it is necessary to relax our assumption that the purchasing power of money is kept constant and let the horizontal axis in Figure 5-2 measure the stock of real balances instead of the stock of nominal money balances.

r_4k_0 or $r_3k_0 + r_{m3}m_3$, where $r_3 = r_{m3}$.[1] In this way the public is induced to hold the satiety stock of money by being offered a real rate of return $(i - \pi)$ equal to the rate of return on material wealth, even if the marginal productivity of money is zero. This interpretation seems to be more illuminating than Johnson's explanation because it directly points to the fact that the community's gains in terms of real income resulting from the introduction of commodity money, replacing commodity money with credit money, and increasing the stock of credit money up to the satiety level are due to the increase in the productivity of capital resulting therefrom, instead of employing the concept of consumers' surplus.

Our next step is to show that the concept of optimality employed in the above discussion --- implying that the satiety stock of money is the optimum stock --- is not consistent with the true concept of optimality based on the principle of utility maximization and to demonstrate the fact that the optimum stock of money is not necessarily the satiety stock.

We note that in the steady state the physical consumption is the difference between the physical output and the amount of capital required to keep the capital-labor ratio constant and that, therefore, the per-capita physical consumption is

(5.1) $$c = y(k,m,b) - nk.$$

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[1]The extra net gain which is the difference between $r_4k_0 (= r_3k_0 + r_{m3}m_3)$ and $r_2k_0 + r_{m2}m_1$ can be considered as identical to the triangular consumers' surplus area m_1PD' in Figure 5-1. Also, notice that $r_4k_0 - (r_2k_0 + r_{m2}m_1)$ consists of the two elements, $r_4k_0 - (r_3k_0 + r_{m3}m_2)$ and $(r_3k_0 + r_{m3}m_2) - (r_2k_0 + r_{m2}m_1)$.

To determine the optimal rate of monetary expansion or the optimal rate of price change we differentiate (5.1) with respect to π and obtain

$$(5.2)\quad dc/d\pi = (y_k - n)dk/d\pi + y_m dm/d\pi + y_b db/d\pi.$$

Since real balances and real bonds, when they are treated as factors of production, contribute to the physical output through the production function, they do not directly enter the utility function but increase the total utility indirectly by increasing the physical output. Therefore, when the maximum physical consumption is achieved the total utility is also maximized.[1] The maximum physical consumption is achieved and hence the total utility is maximized when $dc/d\pi = 0$. Hence, the optimality condition (5.3) is derived by setting (5.2) equal to zero.

$$(5.3)\quad (y_k - n)dk/d\pi + y_m dm/d\pi + y_b db/d\pi = 0.$$

where y_m is equal to y_b. Equation (5.3) represents the true optimality condition in the monetary economy. From (5.3) it is clear that the Golden Rule condition is not necessarily an optimal condition unless $dm/d\pi$ and $db/d\pi$ are both equal to zero or $y_m = y_b$ is equal to zero. Since $dm/d\pi$ and $db/d\pi$ are not equal to zero but less than zero as shown in Chapter III, the Golden Rule condition will be an optimality condition only when $y_m = y_b$ is equal to zero, that is, only when the stock of money and bonds are at their satiety level. However, the optimum does not necessarily require the satiety stock of monetary assets because the total utility can be maximized at any level of the stock of monetary

[1]Of course, the per capita consumption per unit of time will be affected by the rate of capital accumulation. Therefore, the rate of capital accumulation must be such that the per capita consumption per unit of time is maximized. In a non-monetary model this is achieved when the rate of capital accumulation is such that the rate of return on capital is equal to the natural rate of growth (Golden Rule). In a monetary model, this is achieved when (5.3) is satisfied.

assets as long as the condition (5.3) is satisfied. Since it has been shown in Chapter III that in the steady state $dk/d\pi$ is positive, and $dm/d\pi$ and $db/d\pi$ are negative, it follows that, at a less-than-satiety stock of money (and/or bonds), i.e., when $y_m = y_b$ is greater than zero, the optimum requires $y_k > n$. Therefore, when money and bonds are at their less-than-satiety stock, the Golden Rule condition can no longer be an optimality condition. In other words, the optimum requires (1) that $y_k = n$ (Golden Rule) when $y_m = y_b = 0$ and (2) that $y_k > n$ when $y_m = y_b > 0$.

However, in the steady state, $n = \mu - \pi$. Therefore, in the first case in which the optimum requires $y_k = n$ (Golden Rule), the optimal rate of monetary expansion is found to be $\mu = y_k + \pi = y_m = y_b = 0$, which is the marginal productivity of real balances (or the marginal productivity of real bonds). Thus, when monetary assets are at their satiety level, the optimal rate of monetary expansion is equal to zero. In the second case in which the optimum requires $y_k > n$, the optimal rate of monetary expansion must be less than $y_k + \pi$, i.e., $\mu < y_k + \pi = y_m = y_b > 0$. Thus, when monetary assets are at their less-than-satiety level, the optimal rate of monetary expansion is less than the marginal productivity of monetary assets which is positive.

Then, would it be possible to make a general statement consistent with both cases mentioned above? To answer this question we derive a general relationship (5.4) by substituting the expressions for $dk/d\pi$, $dm/d\pi$ and $db/d\pi$, that is, (3.18) - (3.20), derived in Chapter III, into our true optimality condition (5.2).[1]

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[1]For mathematical derivations, see Appendix VI.

(5.4) $y_k - y_m = n.$

Equation (5.4) implies that in the optimum state the marginal productivity of capital exceeds the marginal productivity of money balances (or bonds) by an amount equal to the rate of growth of effective labor (or the natural rate of growth).[1] The factor market equilibrium condition, $y_k = y_m - \pi$, then implies that

(5.5) $n = -\pi$

which again implies that $\mu = 0$, since $\mu - n - \pi = 0$ in the steady state. In other words, the optimal rate of monetary expansion is equal to zero regardless of whether the satiety stock of monetary assets has been achieved or not. This result coincides with our previous results of two cases because the zero rate of monetary expansion is certainly less than a positive marginal productivity of monetary assets.

Therefore, conditions (5.4) and (5.5) represent the true and general optimality condition and the optimal rate of growth of money is equal to zero whatever the level of monetary stock may be, as long as the condition (5.4) is satisfied. The optimum monetary policy would be to keep the stock of monetary assets constant at the existing level, once the optimum stock has been attained, and let the general price level decrease at the natural rate of growth. Of course, this is true only when π (or equivalently μ) is chosen as the only policy parameter. Different results may be obtained if either a variable saving ratio or endogenously determined technical progress function is introduced.

[1]The fact that the marginal productivities of money and capital are different does not imply that there is a disequilibrium in the factor market, because their real rates of return are still the same, i.e., $y_k = y_m - \pi = y_b - \pi$.

However, if the interest payment on money is allowed, the optimal rate of monetary expansion will no longer be equal to zero. Suppose that the nominal rate of interest on money, i, is instituted. Assume that the fund needed to pay the interest is financed by a costless creation of credit money. Then, the factor market equilibrium condition will be represented by

(5.6) $y_k = y_m + i - \pi$

which implies that the optimal rate of monetary expansion is equal to the nominal rate of interest on money.[1] Now assume that the fund needed to pay the interest on money comes from the increased productivity of capital made possible by the increase in the stock of money in a manner that has been described in the earlier part of this chapter. (pp. 72-77). Then, the factor market equilibrium condition becomes

(5.7) $y_k - i = y_m + i - \pi$

which implies that the optimal rate of growth of money is twice the nominal rate of interest on money.[2]

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[1] $y_k = y_m + i - \pi$
$y_k - y_m = i - \pi$
$n = i - \pi$ via (5.4)
$i = n + \pi$
$\therefore \mu = i$, since $\mu - \pi - n = 0$.

[2] $y_k - i = y_m + i - \pi$
$y_k - y_m = 2i - \pi$
$n = 2i - \pi$ via (5.4)
$n + \pi = 2i$
$\therefore \mu = 2i$, since $\mu - \pi - n = 0$.

Finally we confirm that the results obtained for $dk/d\pi$, $dm/d\pi$ and $db/d\pi$ in Chapter III are also true in the optimum state. Substituting (5.4) into (5.3) yields

(5.8) $dk/d\pi + dm/d\pi + db/d\pi = 0$

which implies that the extent to which k changes is exactly the same as the extent to which both m and b change in the opposite direction as a result of a change in the rate of monetary expansion, i.e., $dk/d\pi = -(dm/d\pi + db/d\pi)$. From the factor market equilibrium conditions we derive two more equations[1] expressed in terms of $dk/d\pi$, $dm/d\pi$ and $db/d\pi$. These two equations together with (5.8) constitute a system of three simultaneous equations which can be solved for $dk/d\pi$, $dm/d\pi$ and $db/d\pi$. The signs of these comparative-dynamic derivatives are found to be $dk/d\pi > 0$, $dm/d\pi < 0$ and $db/d\pi < 0$.[2] In other words, an increase in the rate of monetary expansion will result in an increase in the equilibrium capital intensity and a decrease in the equilibrium values of per capita real balances and per capita real bonds. Hence the results obtained in Chapter III concerning the comparative-dynamic derivatives are also true in the case of the optimum state. This must be so because an optimum state is also a steady state although it does not necessarily follow that a steady state is also an optimum state.

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[1]Differentiate $y_k = y_m - \pi$ and $y_k = y_b - \pi$, which represent the factor market equilibrium conditions, and obtain

$$(y_{kk} - y_{mk})dk/d\pi + (y_{km} - y_{mm})dm/d\pi + (y_{kb} - y_{mb})db/d\pi = -1$$
$$(y_{kk} - y_{bk})dk/d\pi + (y_{km} - y_{bm})dm/d\pi + (y_{kb} - y_{bb})db/d\pi = -1.$$

[2]See mathematical appendix, Appendix VI.

The Case in Which Bonds Market is Non-Existent

If we assumed that there were only three markets --- labor, capital and money markets --- and that money were the only type of financial asset, we would have a production function of the form $y = y(k,m)$ and the optimum condition of the form

(5.9) $(y_k - n)dk/d\pi + y_m dm/d\pi = 0$

from which the conditions for an optimum, $y_k - y_m = n$ and $n = -\pi$, can be derived. The system of simultaneous equations would now consist of only two equations:[1]

(5.10) $dk/d\pi + dm/d\pi = 0$

(5.11) $(y_{mk} - y_{kk})dk/d\pi + (y_{mm} - y_{km})dm/d\pi = 1$

whose solutions can be found graphically. Two linear equations, (5.10) and (5.11), are drawn in Figure 5-3. Straight line A

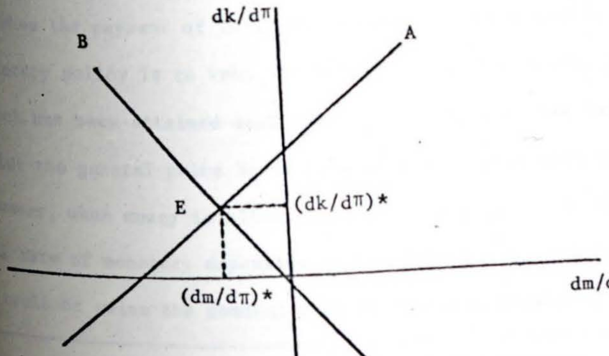

Figure 5-3.--Determination of the Signs of Comparative-Dynamic Derivatives in the Optimum State

[1]Equation (5.10) is obtained by substituting $y_k - y_m = n$ into (5.9) and equation (5.11) is obtained by differentiating $y_k = y_m - \pi$ with respect to π, where y_k and y_m are functions of k and m only.

represents equation (5.11).[1] Straight line B represents equation (5.10). The point E satisfies (5.10) and (5.11) simultaneously and $dk/d\pi$ is positive and $dm/d\pi$ is negative at this point. Therefore, it can be concluded that in the optimum state $dk/d\pi > 0$ and $dm/d\pi < 0$.

To summarize our discussion in this chapter, we note that the concept of optimality which states the optimum stock of money is reached at its satiety level is not a valid concept because any stock of money, not necessarily the satiety stock, can be an optimum one as long as the relation $y_k - y_m = n$ is satisfied. We have also shown that the optimal steady-state rate of growth of money is zero regardless of whether the optimum stock is at the satiety level or not, when the payment of interest on money is not allowed. This result has been obtained under the assumptions that the population growth and technological progress are exogenously determined, that the saving ratio is constant, and that, therefore, the rate of monetary expansion is the only policy parameter available. Therefore, when the payment of interest on money is not allowed, the optimal monetary policy is to keep the stock of money constant, once the optimum stock has been attained such that $y_k - y_m = n$, at the existing level and let the general price level fall at the natural rate of growth.

However, when money is allowed to bear a nominal rate of interest, the optimal rate of monetary expansion is equal to the nominal rate of interest itself or twice the nominal rate of interest depending upon

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[1]Equation (5.11) can be rearranged as
$dk/d\pi = \{(y_{mm} - y_{km})/(y_{kk} - y_{mk})\}\ dm/d\pi + 1/(y_{mk} - y_{kk})$ where the coefficient of $dm/d\pi$ and the constant term are both positive, since $y_{mm} < 0$, $y_{kk} < 0$, and $y_{km} = y_{mk} > 0$.

whether the payment of interest is financed by the creation of credit money or by transferring some part of yields on capital.

Finally, it has been shown that in the optimum state an increase in the rate of monetary expansion would result in an increase in the equilibrium capital intensity and in a decrease in the equilibrium values of per-capita real balances and per-capita real bonds.

CHAPTER VI

CONCLUSIONS

Our discussion has been centered around two of the three fundamental questions in the theory of monetary economic growth, that is, the comparative-dynamic and optimality aspects of growth equilibrium. Specifically the questions we have been concerned with have been: (1) can variations in the rate of monetary expansion affect the time profiles of the real variables and, hence, the equilibrium values of real variables, and (2) is there an optimum growth of money? The question of optimal degree of financial intermediation has not been dealt with in depth in this thesis. In Chapter I, the monetary growth models of Tobin, Levhari-Patinkin, Sidrauski, and Stein have been analyzed with regard to these questions.

In Tobin's outside-money model, where money is considered as stores of values, the equilibrium capital intensity is declared to be lower than what would have been in the non-monetary growth model. This has been a cause of serious confusion among the economists who fail to distinguish between the barter model and non-monetary model. Tobin contends that an increase in the rate of monetary expansion increases the equilibrium capital intensity and, hence, the Harrodian impasse can be removed by an appropriate government monetary policy. This monetary policy can be carried out by varying the nominal rate of interest on money (i) and the rate of monetary expansion (μ). However, Johnson maintains that an increase in the rate of monetary expansion can either increase or decrease the equilibrium capital intensity depending upon whether or not it lowers the proportion of any given amount of savings, which has to be invested in the accumulation of real capital equipment, to a greater extent than it lowers the utility yield on money balances and, hence, lowers the proportion of national income available for capital accumulation.

Levhari and Patinkin use the two different approaches (money as a consumer's good approach and money as a producer's good approach) to analyze some of the characteristics of the monetary growth model. They fail to determine whether an increase in the rate of monetary expansion will increase or decrease the equilibrium capital intensity. However, they maintain that when the consumer's good approach is used a sufficiently large increase in the rate of monetary expansion will have to increase the equilibrium capital intensity. On the other hand, using the utility-maximizing approach, it is said that the equilibrium capital intensity is unaffected by the change in the rate of monetary expansion when the assumption is made that the subjective rate of time preference is independent of the size of real balances per unit of effective labor. This result is identical with the Sidrauski's result on this topic.

According to Sidrauski, "the long-run capital stock of economy is independent of the rate of monetary expansion, although in the short-run an increase in the rate of monetary expansion reduces the rate of capital accumulation." This result is due to the assumption that the subjective rate of time preference is constant and unaffected by the stock of nonhuman wealth and its composition. "A rise in the rate of monetary expansion results in an equal absolute increase in the rate of change in prices; it reduces the stock of real cash but it does not affect steady-state consumption. It therefore follows that the higher the rate of monetary expansion the lower will be the steady-state level of utility."[1]

Stein's "Keynes-Wicksell" model fails to determine whether the effects of a change in the rate of monetary expansion on the equilibrium values of real variables are positive or negative. Stein concludes that the steady-state values of real variables can either fall, rise, or remain constant as a result of a rise in the rate of monetary expansion.

As for the optimum growth of money, Tobin maintains in his outside-money model that it is not optimal to have the real rate of interest lower than the natural rate of growth. In other words, the optimum requires that $n \leq i - \mu + n$ in the steady-state, i.e., that the rate of monetary expansion must not exceed the nominal rate of interest on money ($\mu \leq i$). If we have the real rate of interest equal to the natural rate (Golden Rule), then the rate of monetary expansion will be equal to the nominal rate of interest on money (i.e., $\mu = i$), and the rate of price change will be equal to $i-n$ in the steady state. If the price stability ($\pi = 0$) is desired, then we should have $i = \mu = n$. Otherwise, there will be either inflation or deflation. However, since in equilibrium the steady rate of price change is wholly anticipated, and one rate is as good as another, we cannot establish any criterion for choosing a particular common values of μ and i on optimality grounds. On the other hand, if in the absense of government debt (outside money) the equilibrium marginal productivity of capital would exceed n, it is not optimal to absorb any saving in government debt. Tobin states:

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[1]Sidrauski, "Rational Choice and Patterns of Growth", JPE, 77, No. 4, Part II (July/August, 1969), p. 585.

> The general conclusion is that there is an optimal rate of growth of the supply of outside money, equal to or smaller than the nominal interest rate i, only to the extent that diversion of saving into this vehicle is necessary to keep the marginal productivity of capital from falling below n. In the absence of such a tendency for over-saving, it is not optimal to absorb any saving in outside money, or deadweight debt.[1]

Tobin also considers money as a means of payment and argues that the means of payment should bear a high enough real rate of return to remove the incentive to economize them without answering the question of what the optimal rate of growth of means of payments is.

Levhari and Patinkin consider the problem of optimum growth of money and obtain the condition for maximum constant utility per unit of time as follows:

$$(y_k - n)dk/d\pi + y_m dm/d\pi = 0$$

which implies that the rate of price change π that satisfies the optimum condition is not in general the one that generates the equilibrium capital intensity specified by the Golden Rule since $\frac{dm}{d\pi}$ is not zero in general.

The present writer has, in this volume, tried to make a number of contributions to the theory of monetary economic growth. To this end, the nonhuman wealth is separated into three broad categories, namely, capital, money, and bonds, and they are all treated as factors of production along with labor, because money and bonds increase the efficiency of the economy by releasing real resources which would have been tied up

[1]Tobin, "Notes on Optimal Monetary Growth", JPE, 74 (Supplement, No. 4, Part II, July/August, 1968), p. 841.

in arranging barter transactions or by performing the functions of financial intermediaries between savers and investors. Both inside and outside assets are considered a part of community's wealth. A model economy is constructed under these new assumptions in Chapter II.

There are a number of significant departures from the conventional models in the model presented in this thesis. First of all, the production function includes real bonds (bonds in real terms) as a factor of production in addition to real balances. Secondly, money and bonds include inside money and bonds respectively as well as outside money and bonds, and they are considered a part of community's wealth at the same time they are factors of production. Therefore, the definition of real disposable income is different from both Tobin's and Levhari-Patinkin's. Both equilibrium and disequilibrium approaches have been used in developing a dynamic system which characterizes the time profiles of real variables. In equilibrium approach all markets have been assumed to be in equilibrium. In disequilibrium approach it has been assumed that money and commodity markets are normally out of equilibrium and that the existence of excess demands in these markets activates price changes and causes the actual capital formation to deviate from the planned investment or planned saving. Therefore, another characteristic of this model is that the independent investment function is postulated in Chapter IV.

The thesis has been concerned with a differential equations structure. Given the initial values of endogenous variables and the exogenously determined rates of population growth and technological progress, the time profiles of endogenous variables expressed in real per capita terms have been found. Since there are three endogenous per capita real variables k, m, and b in the model, three differential equations describing the time profiles of k, m, and b respectively are required for a complete dynamic system. Two different dynamic systems have been developed in this thesis --- one in Chapter III, using an equilibrium approach, and the other in Chapter IV, using a disequilibrium approach.

From these dynamic systems the steady-state (growth equilibrium) solutions have been obtained, and the first two of the three fundamental questions, namely, the comparative-dynamic and optimality questions have been examined. In Chapter IV, where a disequilibrium method is used, no attempt has been made to discuss optimality questions because of the highly complex nature of the disequilibrium model. The optimality question of the equilibrium model introduced in Chapter III has been dealt with in Chapter V.

In what follows, the important results obtained in this volume are summarized. Using equilibrium approach the following results have been obtained in Chapters III and V:

a). The equilibrium capital intensity in a model economy such as the one introduced in Chapter II is lower than what would have been in a non-monetary model such as that of Solow's. This result coincides with Tobin's but not for the identical reason. In the present model the equilibrium capital intensity is lower in the monetary model, even if both contributing effect (output effect) and negative effect (leakage effect) of real money balances and bonds are taken into consideration, whereas in Tobin model only the leakage effect is considered.

b). An increase in the rate of monetary expansion will result in a rise in the equilibrium capital intensity and a decrease in the equilibrium values of per capita real balances and real bonds so long as a small amount increase in one factor of production increases the marginal productivities of the remaining factors at the same rate. This is a significant conclusion because there has yet been no definite conclusion on this question of comparative dynamics.[1]

c). The optimum stock of money or bonds is not necessarily the satiety stock but can be any level of the stock as long as the relation $y_k - y_m = n$ is satisfied, i.e., as long as the marginal productivity of capital exceeds that of real balances by an amount equal to the natural rate of growth.[2] Therefore, there is no reason why the public must be induced to hold the satiety stock of money by means of the interest payment on money. In the special case in which money and bonds are at their satiety level, the Golden Rule condition is also an optimum condition and, hence, the optimum requires the marginal productivity of capital to be equal to the natural rate of growth.

d). The optimal rate of growth of money is the rate which keeps the stock of money at its optimum level. In general, the optimal rate of growth of money is equal to zero regardless of whether the stock of money and bonds are at their satiety level or not. However, when the

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[1]Tobin first presented this proposition, but the proposition was not accompanied by a proof, verbal or mathematical. Tobin simply assumed that the Pigou effect will eventually be stronger than the Wicksell effect.

[2]Of course, the relationship $y_k - y_m = n$ does not imply that the real rates of return on capital and money are different because it is still true that $y_k = y_m - \pi$ in this relationship.

nominal rate of interest i is paid on money, the optimal rate of growth of money is equal to the nominal rate of interest itself, again regardless of whether the stock of money and bonds are at their satiety levels or not, provided that the interest payment is financed by the costless creation of paper money, inside or outside, rather than by transferring some of the yields on capital.[1]

The significance of this conclusion is that it contains the elements of both Tobin's and Levhari-Patinkin's results, and yet different in its nature. In the case in which the interest is not paid on money, the Levhari-Patinkin's conclusion remains the same in spite of the fact that another factor of production, bond, is added to the Levhari-Patinkin production function. In the case in which an interest is paid on money, the conclusion obtained in Chapter V is similar to, but not exactly same as, Tobin's conclusion[2] in so far as the payment of interest on money is financed by the creation of credit money. However, if the payment of interest comes from the increased productivity of capital made possible by the increase in the stock of real balances, the optimal rate of growth of money is twice the nominal interest rate.

Using disequilibrium approach the following conclusions have been obtained.

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[1] If the payment of interest on money were to be financed by transferring some of the yields on capital, then the optimal rate of growth of money would be twice the nominal rate of interest on money itself.

[2] Tobin's conclusion is that "there is an optimal rate of growth of the supply of outside money, equal to or smaller than the nominal interest rate, i, only to the extent that diversion of saving into this vehicle is necessary to keep the marginal productivity of capital from falling below n". Tobin, "Notes on Optimal Monetary Growth", p. 841.

e). No definite statement can be made on whether an increase in the rate of monetary expansion will increase or decrease the equilibrium values of real variables. However, in this approach it has been found that the direction of the effect of a change in the rate of monetary expansion on the equilibrium values of real variables is dependent upon the numerical values of α and β, i.e., upon the institutional market arrangements which determine how output is distributed between savers and investors in the periods of rising prices and the speed of market response in eliminating excess demands.

f). Disequilibrium models are unattractive because the results depend heavily on the numerical values of α and β which cannot easily be determined.

No serious attempt has been made in the thesis to discuss the important question of the optimal degree of financial intermediation, not because the present author considers it the least important but because answering this question at the present stage of development of economic theory is considered beyond his intellectual ability.

MATHEMATICAL APPENDIX

NOTATIONS

A Stock of nonhuman wealth

a Per capita stock of nonhuman wealth

b Per capita real stock of bonds

c Per capita physical consumption

$D = d/dt =$ dot ($\cdot$) over the symbol

E Aggregate expenditure

I Investment

i Nominal rate of interest

K Capital stock

k Capital-labor ratio (capital intensity)

$\hat{k}$ Capital-output ratio

L Labor force measured in efficiency unit (effective labor)

M Nominal stock of money

m Per capita real money balances (M/PL)

$\hat{m}$ Real money balances per unit of capital (M/PK)

n Rate of growth of effective labor (natural rate of growth)

P General price level

r Rate of return on capital (Marginal productivity of capital)

r_m Real rate of return on money

S Saving

S_p Physical saving

s Overall savings ratio

s^* Per capita saving

t Time

U Utility

u Rate of depreciation

v Real per capita net government transfer ($\dot{M}/PL$)

W Total welfare

w Wealth-income ratio

x Labor-capital ratio ($1/k$)

Y Physical output

Y^* Real disposable income

y Per capita physical output

$\hat{y}$ Physical output per unit of capital

z Real stock of monetary assets ($M/P + B/P$)

α Coefficient representing the institutional framework which determines the extent to which consumer and producer demands are satisfied when there is an excess aggregate demand.

β Coefficient representing the speed of market adjustments

γ Rate of growth of the nominal stock of bonds

δ Subjective rate of time preference

θ Ratio of outside to total stock of money

μ Rate of growth of the nominal stock of money (rate of monetary expansion)

π Actual rate of price change

π^* Expected rate of price change

ρ Factor of proportionality relating real balances and physical output (income velocity)

σ Physical savings ratio

APPENDIX I. SOLOW MODEL

(1.1) $Y = Y(K,L)$ production function

(1.2) $\dot{K} = sY$ saving-investment identity

(1.3) $\dot{K} = sY(K,L)$ from (1.1) and (1.2)

(1.4) $L = L_o e^{nt}$ exponential growth of labor force

(1.5) $\dot{K} = sY(K,L_o e^{nt})$ from (1.3) and (1.4)

(1.6) $K = kL = kL_o e^{nt}$; $k = K/L$

Differentiating (1.6) with respect to time yields

(1.7) $\dot{K} = L_o e^{nt}\dot{k} + nkL_o e^{nt} = (\dot{k} + nk)L_o e^{nt}$

(1.8) $(\dot{k} + nk)\ L_o e^{nt} = sY(K,L_o e^{nt}) = sL_o e^{nt}Y(k,1)$ from (1.5) and (1.7)

Hence,

(1.9) $\dot{k} + nk = sY(k,1) = sy(k)$

which is equivalent to

(1.10) $\dot{k} = sy(k) - nk.$

In the steady state,

(1.11) $nk = sy(k).$

Hence, the equilibrium capital intensity k^* is

(1.12) $k^* = sy(k)/n.$

APPENDIX II. TOBIN MODEL

(2.1) $Y = Y(K,L)$ production function

(2.2) $Y^* = Y + D(M/P) = Y + (\mu - \pi)M/P$ definition of real disposable income

(2.3) $S = sY^* = s[Y + (\mu - \pi)M/P]$ saving hypothesis

(2.4) $S = \dot{A} = \dot{K} + D(M/P) = \dot{K} + (\mu - \pi)M/P$

(2.5) $\dot{K} = sY^* - (\mu - \pi)M/P = sY - (1-s)(\mu - \pi)M/P$ from (2.3) and (2.4)

(2.6) $\dot{K} = L\dot{k} + nkL$, since $K = kL$. (see 1.7)

(2.7) $\dot{k} = sy(k) - (1-s)(\mu - \pi)m - nk$ from (2.5) and (2.6)

In the steady state, $n = \mu - \pi$. Hence,

(2.8) $nk = sy(k) - (1-s)nm.$

The equilibrium capital intensity k^* is

(2.9) $k^* = sy(k)/n - (1-s)m.$

APPENDIX III. LEVHARI-PATINKIN MODEL

(a) Money as a Consumer's Good

(3.a.1) $Y = Y(K,L)$ production function

(3.a.2) $Y^* = Y + (\mu - \pi)M/P + (r + \pi)M/P = Y + (\mu + r)M/P$ real disposable income

(3.a.3) $S = sY^* = s\,[Y + (\mu + r)M/P]$ saving hypothesis (see 2.3)

(3.a.4) $S = \dot{A} = \dot{K} + (\mu - \pi)M/P$ (see 2.4)

(3.a.5) $\dot{K} = sY - (1-s)(\mu - \pi)M/P + s(r + \pi)M/P$ from (3.a.3) and (3.a.4)

(3.a.6) $\dot{K} = \dot{k}L + nkL$ (see 2.6)

(3.a.7) $\dot{k} = sy(k) - (1-s)(\mu - \pi)m + s(r + \pi)m - nk$ from (3.a.5) and (3.a.6)

Demand for real balances is a function of physical output and money market is always in equilibrium so that

(3.a.8) $m = \rho y(k)$, where $\rho = \rho\,(y'(k) + \pi)$; $\rho' < 0$.

(3.a.9) $\dot{k} = [s - (1-s)(\mu - \pi)\rho + s(r + \pi)\rho]y(k) - nk$ from (3.a.7) and (3.a.8)

In the steady state,

(3.a.10) $nk = [s - (1-s)n\rho + s(r + \pi)\rho]y(k)$.

Therefore, the equilibrium capital intensity k^* is

(3.a.11) $k^* = y(k)/n\{s[1 + \rho(n + r + \pi)] - \rho n\}$.

(3.a.12) $S_p = S - D(M/P)$ definition of physical saving

(3.a.13) $S_p = s[Y + (\mu + r)M/P] - (\mu - \pi)M/P$ from (3.a.3) and (3.a.12)

(3.a.14) $S_p = sY[1 + \rho(n + r + \pi)] - \rho nY$ from (3.a.8 and (3.a.13)

Hence,

(3.a.15) $\sigma = S_p/Y = s[1 + \rho(n + r + \pi)] - \rho n$.

(3.a.16) $k^* = (\sigma/n)y(k)$ from (3.a.11) and (3.a.15)

Equation (3.a.16) can be rewritten as

(3.a.17) $y(k)/k = g(k) = n/\sigma(k,\pi)$, since both ρ and r are a function of k.

Differentiating (3.a.17) with respect to π yields

$$(3.a.18)\quad dk/d\pi = \frac{\sigma_\pi}{-(\sigma^2/n)g'(k) - \sigma_k}$$

whose sign is ambiguous.

Differentiating (3.a.8) with respect to π yields

$$(3.a.19)\quad dm/d\pi = \rho'[y''(k)dk/d\pi + 1]y(k) + \rho\, y'(k)dk/d\pi$$

whose sign is also ambiguous.

Per capita physical consumption is

$$(3.a.20)\quad c(k) = y(k) - \sigma\, y(k)$$

which, in the steady state, reduces to

$$(3.a.21)\quad c(k) = y(k) - nk. \quad \text{(see 3.a.16)}$$

$$(3.a.22)\quad U = U(c,m) = U[y(k) - nk,\ m] \quad \text{utility function}$$

The necessary condition for utility maximization is

$$(3.a.23)\quad dU = U_1 \cdot [y'(k) - n]dk + U_2\ dm = 0$$

which will be satisfied if

$$(3.a.24)\quad y'(k) - n = 0$$

and

$$(3.a.25)\quad U_2\ [y(k) - nk, m] = 0$$

simultaneously, assuming non-satiety of commodity consumption, i.e., $U_1 \neq 0$.

$$(3.a.26)\quad i = y'(k) + \pi = 0 \quad \text{due to the assumption that money is non-interest-bearing.}$$

$$(3.a.27)\quad n = y'(k) = -\pi \quad \text{from (3.a.24) and (3.a.26)}$$

In the steady state,

$$(3.a.28)\quad \mu = \pi + n = 0$$

which implies that the optimal rate of growth of money is zero.

It follows from (3.a.23) that

$$(3.a.29)\quad dU/d\pi = U_1 \cdot [y'(k) - n]dk/d\pi + U_2\ dm/d\pi = 0$$

which implies that the Golden Rule capital intensity is not necessarily the optimal capital intensity and that the optimum stock of money is not necessarily the satiety stock.

(b) Money as a Producer's Good

(3.b.1) $Y = Y(K,L,M/P)$ production function

(3.b.2) $y = y(k,m)$

(3.b.3) $Y^* = Y + (\mu - \pi)M/P$ definition of disposable income (same as Tobin's)

(3.b.4) $S = sY^* = s[Y + (\mu - \pi)M/P]$ saving hypothesis

(3.b.5) $S = \dot{K} + (\mu - \pi)M/P$

(3.b.6) $\dot{K} = sY - (1-s)(\mu - \pi)M/P$ from (3.b.4) and (3.b.5)

(3.b.7) $\dot{K} = L\dot{k} + nkL$, since $K = kL$.

(3.b.8) $\dot{k} = sy(k,m) - (1 - s)(\mu - \pi)m - nk$ from (3.b.6) and (3.b.7)

In the steady state,

(3.b.9) $nk = sy(k,m) - (1 - s)(mn)$.

Hence, the equilibrium capital intensity k^* is

(3.b.10) $k^* = sy(k,m)/n - (1 - s)m$.

Factor market equilibrium condition is

(3.b.11) $y_k(k,m) = y_m(k,m) - \pi$

from which the demand function for real balances is derived.

(3.b.12) $m = m(k,\pi)$. Hence,

(3.b.13) $k^* = (s/n)y[k,m(k,\pi)]-(1 - s)m(k,\pi)$ from (3.b.10) and (3.b.12)

which may be rewritten as

(3.b.14) $k^* = (1/n)\{s - (1 - s)nm(k,\pi)/y[k,m(k,\pi)]\}y[k,m(k,\pi)]$

$= (1/n)\sigma(k,\pi)y[k,m(k,\pi)]$

where $\sigma(k,\pi) = \{s - (1 - s)nm(k,\pi)/y[k,m(k,\pi)]\}$ is a physical savings ratio.

Hence,

(3.b.15) $k^* = \sigma(k,\pi)y(k,m)/n$.

Implicitly differentiating (3.b.8) and (3.b.9) with respect to π yields

(3.b.16) $[sy_k - n]dk/d\pi + [sy_m + (s - 1)n]dm/d\pi = 0$

and

(3.b.17) $[y_{mk} - y_{kk}]dk/d\pi + [y_{mm} - y_{km}]dm/d\pi = 1$

respectively.

Solving (3.b.16) and (3.b.17) simultaneously, using Cramer's rule, yields

(3.b.18) $dk/d\pi = [sy_m + (s - 1)n]/-\Delta$

(3.b.19) $dm/d\pi = (sy_k - n)/\Delta$

where determinant Δ is equal to $[sy_k - n][y_{mm} - y_{km}] -$

$[sy_m + (s - 1)n][y_{mk} - y_{kk}]$.

Per capita physical consumption in the steady state is

(3.b.20) $c = y(k,m) - nk$

from which the condition for maximum steady-state consumption is obtained:

(3.b.21) $dc/d\pi = y_k\ (dk/d\pi) + y_m(dm/d\pi) - n(dk/d\pi) = 0$

which implies that the optimum stock of money is not necessarily the satiety stock.

Substituting (3.b.18) and (3.b.19) into (3.b.21) yields

(3.b.22) $y_k - y_m = n.$

Then, (3.b.9) implies

(3.b.23) $n = -\pi.$

Therefore,

(3.b.24) $\mu = n + \pi = 0$

which implies that the optimum requires a constant stock of money.

APPENDIX IV. SIDRAUSKI MODEL

(4.1) $U_t = U(c_t, m_t)$ time invariant utility function

(4.2) $W = \int_0^\infty [U(c_t, m_t)] e^{-\delta t} dt$ total welfare function to be maximized

(4.3) $y_t = y(k_t)$ production function

(4.4) $y(k_t) + v_t = c_t + s_t^*$ real disposable income

Assume that net transfer payments are entirely financed by the creation of government non-interest-bearing debt, outside money. Then

(4.5) $v_t = \dot{M}_t/(P_t L_t) = \mu\, m_t.$

(4.6) $s_t^* = \dot{k}_t + (u + n)k_t + (\pi^*_t + n)m_t + \dot{m}_t$ disposition of per capita gross saving

(4.7) $y(k_t) + v_t - (u + n)k_t - (\pi_t^* + n)m_t - \dot{k}_t - \dot{m}_t - c_t = 0$

from (4.4) and (4.5)

(4.8) $a_t = k_t + m_t$

(4.9) $\dot{a}_t = \dot{k}_t + \dot{m}_t$

(4.10) $\dot{a}_t = y(k_t) + v_t - (u + n)k_t - (\pi_t^* + n)\, m_t - c_t$ from (4.7) and (4.9)

Hence, the Lagrange function is

(4.11) $I = \int_0^\infty \{U(c_t, m_t) - \lambda_t [y(k_t) + v_t - (\pi_t^* + n)m_t - (u + n)k_t - c_t - \dot{a}_t] - q_t(a_t - k_t - m_t)\}\, e^{-\delta t} dt.$

The conditions for a maximum total welfare are represented by (4.12)-(4.16):

(4.12) $U_c(c_t, m_t) = \lambda_t$

(4.13) $U_m(c_t, m_t) = \lambda_t(\pi_t^* + \hat{r}_t + n);\ \hat{r}_t = q_t/\lambda_t$

(4.14) $y'(k_t) - (u + n) = \hat{r}_t$

(4.15) $\dot{\lambda}_t/\lambda_t = \delta - \hat{r}_t$

(4.16) $\lim_{t\to\infty} a_t \lambda_t e^{-\delta t} = 0,$

where λ_t and $\hat{r}_t$ can be considered as the implicit price of consumption and the implicit interest rate respectively.

Given the initial value, a_o, and the values of u, n,$\pi^{\star}_{t}$, and v_t, a system of six equations — (4.12)-(4.15) plus two constraints (4.8) and (4.10) — can be solved for the time paths of the six endogenous variables, c_t, m_t, k_t, a_t, λ_t and $\hat{r}_t$, which must also satisfy the transversality condition (4.16).

From (4.12) and (4.13) we derive the demand functions for consumption and real balances:

(4.17) $c = c^o(\lambda,\hat{r},m)$

(4.18) $m = m^o(\lambda,\hat{r},\pi^*)$.

From (4.14) we derive the demand function for real capital:

(4.19) $k = k^o(\hat{r})$.

Hence we have, from (4.8), (4.18) and (4.19),

(4.20) $a = k^o(\hat{r}) + m^o(\lambda,\hat{r},\pi^*)$

from which (4.21) is derived:

(4.21) $\hat{r} = \hat{r}(a,\lambda,\pi^*)$.

Now the demand functions become:

(4.22) $c = c'(a,\lambda,\pi^*)$

(4.23) $m = m'(a,\lambda,\pi^*)$

(4.24) $k = k'(a,\lambda,\pi^*)$.

The solution to a pair of differential equations (4.10) and (4.15) is a local saddle point. Therefore, for given values of a_o, π^* and v, there is only one time path associated with this solution $(a^*,\lambda^*)^o$. Hence we may write

(4.25) $\lambda = \lambda(a,\pi^*,v)$

which can be substituted into (4.22) - (4.24) to obtain the demand functions:

(4.26) $c = c(a,\pi^*,v)$

(4.27) $m = m(a,\pi^*,v)$

(4.28) $k = k(a,\pi^*,v)$.

Sidrauski introduces Cagan's "adaptive expectations" hypothesis which is represented by

(4.29) $\dot{\pi}^* = h(\pi - \pi^*);\ h > 0.$

From (4.5), (4.9), (4.10) and (4.27) we derive

(4.30) $\dot{k} = y(k) - (u + n)k - c(a, \pi^*, v),$

since $\dot{m} = (\mu - \pi - n)m.$

From (4.5), (4.8) and (4.27) we derive

(4.31) $m = m^*(k, \mu, \pi^*)$

and hence

(4.32) $\dot{k} = y(k) - (u + n)k - c^*(k, \mu, \pi^*).$

Differentiating (4.31) with respect to time yields

(4.33) $\mu - \pi - n = (1/m)(\partial m/\partial k \cdot \dot{k} + \partial m/\partial \pi^* \cdot \dot{\pi}^*).$

From (4.29), (4.32) and (4.33) we derive

(4.34) $$\dot{\pi}^* = \frac{1}{1 + (h/m)(\partial m/\partial \pi^*)}\{\mu - \pi^* - n - [y(k) - (u + n)k - c^*(k, \mu, \pi^*)]\}.$$

Differential equations (4.32) and (4.34) constitute a dynamic system which yields the steady-state conditions:

(4.35) $c^* = y(k^*) - (u + n)k^*$

(4.36) $\pi^* = \mu - n.$

Hence, the equilibrium capital intensity k^* is

(4.37) $$k^* = \frac{1}{u + n}[y(k^*) - c^*]$$

which is independent of the rate of monetary expansion μ or the rate of price change π.

APPENDIX V. STEIN MODEL

(5.1) $Y = Y(K,L)$ production function

(5.2) $\hat{y} = \hat{y}(x);\ \hat{y} = Y/K$ and $x = L/K$

The linear-homogeneity of (5.1) enables us to write

(5.3) $r = \hat{y}(x) - x\hat{y}'(x) = r(x);\ r' > 0 \quad (r = \partial Y/\partial K)$

(5.4) $S/K = S^*(\hat{y}, \theta\,\hat{m});\ S_1^* > 0$ and $S_2^* < 0$

where S represents planned saving and θ the ratio of outside to total stock of money. Then it follows from (5.3) and (5.4) that

(5.5) $S/K = S(x,\hat{m});\ S_1 > 0,\ S_2 \leqq 0 \quad (S_2 = 0$ when $\theta = 0)$

Independent investment function is

(5.6) $I/K = n + r(x) + \pi^* - i$

where I is planned investment.

Disequilibrium in the commodity market activates price changes, i.e.

(5.7) $\pi = \beta(I/K - S/K)$

where β represents "the speed of market adjustments." Hence

(5.8) $\pi = \beta[n + r(x) + \pi^* - i - S(x,\hat{m})]$.

Assuming that the growth of capital stock is a linear combination of planned saving and planned investment, when prices change, we have

(5.9) $\dot{K}/K = \alpha(I/K) + (1-\alpha)(S/K);\quad 0 < \alpha < 1$ when $\pi > 0$.

(5.10) $E = C + I = Y + I - S$ Aggregate planned expenditure

(5.11) $E/K = \hat{y} + (I - S)/K = \hat{y} + \pi/\beta$ from (5.7) and (5.10).

The demand for real balances per unit of capital is

(5.12) $L = L[\hat{y}(x) + \pi/\beta\quad r(x) + \pi^*,\ i,\ \theta\hat{m}];\ L_1 > 0,\ L_2 < 0,\ L_3 < 0,$

$0 < L_4\,\theta < 1$

Walras' Law states that the sum of excess demands in all markets must be zero. Stein assumes that the labor market is always in equilibrium and that the speed of response in bonds market is infinite such that the excess demand for bonds is immediately eliminated. Thus the flow excess demand for goods per unit of capital (π/β) must be equal to the flow excess supply of real balances per unit of capital. Assuming that the flow excess supply of real balances per unit of capital is proportional to the stock excess supply, we have

(5.13) $\pi/\beta h = \hat{m} - L[\hat{y}(x) + \pi/\beta,\ r(x) + \pi^*,\ i,\ \theta\hat{m}]$

where $0 < h < \infty$ is the factor of proportionality relating the flow and stock supplies of real balances per unit of capital.

In the steady state,

(5.14) $\pi^* = \mu - n.$

The dynamic system is represented by a pair of differential equations:

(5.15) $\dot{x}/x = n - \dot{K}/K$

(5.16) $\dot{\hat{m}}/\hat{m} = \mu - \pi - \dot{K}/K.$

Solving (5.8) and (5.13) for π and i in terms of x and $\hat{m}$, given exogenous variables, μ and n, yields

(5.17) $\pi = \pi(x,\hat{m};\ \mu,n);\qquad \pi_{\hat{m}} > 0,\ \pi_{\mu} > 0$

(5.18) $i = i(x,\hat{m};\ \mu,n).$

From (5.8), (5.9), (5.15) and (5.17) we derive

(5.19) $\dot{x}/x = n - (\alpha/\beta)\ \pi(x,\hat{m};\ \mu,n) - S(x,\hat{m}) = F(x,\hat{m};\ \mu,n).$

From (5.8), (5.9), (5.16) and (5.17) we derive

(5.20) $\dot{\hat{m}}/\hat{m} = \mu - (1 + \alpha/\beta)\ \pi(x,\hat{m};\ \mu,n) - S(x,\hat{m}) = G(x,\hat{m};\ \mu,n).$

It is assumed that, for a given set of values of μ and n, there exist strictly positive steady-state values, x^* and $\hat{m}^*$, such that $F(x^*,\hat{m}^*;\ \mu,n) = 0$ and $G(x^*,\hat{m}^*;\ \mu,n) = 0$. Solving the characteristic equations for the Taylor approximation to the differential equations (5.19) and (5.20) at $(x^*,\hat{m}^*)$ yields the stability conditions:

(5.21) $X^*F_1 + \hat{m}^*G_2 < 0$

(5.22) $x^*\hat{m}^*(F_1G_2 - F_2G_1) > 0.$

To examine the comparative-dynamic aspects of the equilibrium we derive, from the steady-state conditions, a system of simultaneous equations:

(5.23) $F_1(dx^*/d\mu) + F_2(d\hat{m}^*/d\mu) + F_\mu = 0$

(5.24) $G_1(dx^*/d\mu) + G_2(d\hat{m}^*/d\mu) + G_\mu = 0$

which can be solved for $dx^*/d\mu$ and $d\hat{m}^*/d\mu$ to obtain

(5.25) $dx^*/d\mu = \dfrac{-F_\mu G_2 + F_2 G_\mu}{F_1 G_2 - F_2 G_1}$

(5.26) $d\hat{m}^*/d\mu = \dfrac{-F_1 G_\mu + F_\mu G_1}{F_1 G_2 - F_2 G_1}$

where $F_\mu = -\alpha(\pi_\mu/\beta)$, $G_\mu = 1 - (1 + \alpha/\beta)\pi_\mu$, and $\pi_\mu > 0$.

Stability condition (5.22) requires the denominators of (5.25) and (5.26) to be positive. However, the signs of the numerators are ambiguous. In other words, the effects of changes in the rate of monetary expansion on the equilibrium values of real variables are indeterminate.

The Optimum Stock and Rate of Growth of Money

As far as the optimality aspect of the growth equilibrium is concerned, very little research has been done. As we have noted in the previous section, Sidrauski model "is based on an explicit analysis of individuals' saving behavior, viewed as a process of wealth accumulation aimed at maximizing some intertemporal utility function."[1] However, Sidrauski pays little attention to the optimum stock and rate of growth of money. Instead, he tries to show that along the optimum growth path the equilibrium capital intensity is independent of the rate of monetary expansion.

James Tobin also discusses this question but his argument seems to be based upon the concept of Golden Rule that has been developed on the basis of non-monetary growth model. Tobin argues that "there is an optimal rate of growth of the supply of outside money, equal to or smaller than the nominal interest rate i, only to the extent that diversion of saving into this vehicle is necessary to keep the marginal productivity of capital from falling below n."[2] "In the absence of such a tendency for oversaving, it is not optimal to absorb any saving in outside money or deadweight debt."[3] However, Tobin does not give a rigorous proof to support his argument.

Levhari and Patinkin try to show that the optimal rate of growth of money (outside) is zero and that the optimum stock of money is not

[1]Sidrauski, op. cit., p.575.

[2]Tobin, "Notes on Optimal Monetary Growth," p. 841.

[3]Ibid.

necessarily the satiety stock,[1] whether the money is viewed as a consumer's good or as a producer's good. It is assumed in Levhari-Patinkin model that the rate of inflation is the only policy parameter. Therefore, the optimum requires that the government keep the nominal stock of money constant once the optimum stock of money has been reached. Then the general price level will decrease at the rate equal to the natural rate of growth since in the steady state $n = \mu - \pi$.[2]

The arguments presented above are all concerned with the optimum growth of money interpreted as an alternative store of value. Then what about the optimum growth of money viewed as a means of payment? An important fact is that "means of payment can be supplied either as outside money or as inside money, without affecting in one way or another the optimal supply of saving for capital formation."[3] Economic units try to economize their cash holdings simply because of the scarcity of means of payment. Since there exists assets other than cash, which yield higher real returns, they must make frequent transactions in and out of cash in order to maximize their earnings. This requires a diversion of productive resources into the handling of in-and-out transactions which is socially wasteful. This waste can be avoided by supplying a large enough stock of means of payment to absorb all working balances. "This requires," Tobin says, "that means of payment bear a high enough real

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[1]Levhari and Patinkin consider the optimum growth path as the one with the highest constant level of utility per unit of time.

[2]For mathematical proof, see Appendix III(a)(b).

[3]Tobin, "Notes on Optimal Monetary Growth," p. 843.

rate of return to remove the incentive to economize them."[1] The means of payment could be allowed to bear a nominal interest rate or interest-bearing assets could be allowed to serve as means of payment. For example, checking could be permitted against savings accounts in commercial banks and thrift institutions. Tobin argues that "freeing means of payment from the legal limitation of zero interest would make it theoretically possible to have an efficient growth equilibrium without deflation[2]----- efficient both in the sense that the real rate of interest is high enough to avoid overcapitalization and in the sense that real resources are not diverted into economizing means of payment."[3]

The Optimal Degree of Financial Intermediation

We have seen in the previous sections that some economists have been puzzled with Tobin's argument that the equilibrium capital intensity is lower in the monetary model. How can a monetary economy settle down on a lower equilibrium capital intensity and therefore on a lower per capita output and consumption?[4] We have also seen that this is not a

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[1] Ibid.

[2] Price deflation is an alternative way to remove the incentive to economize means of payment. However, applying this method presents some difficulties because deflation will also contribute to increasing the real rate of returns on other assets denominated in monetary unit of account.

[3] Ibid., p. 846.

[4] A lower capital intensity means a lower per capita output and consumption, since $y = y(k)$ where $y'(k) > 0$. This is not necessarily true if the production function is of the form $y = y(k,m)$ and if a lower capital intensity is accompanied by a higher per capital real balances.

valid question to ask because what Tobin really tries to show is that the monetary asset offers an alternative to real capital as a store of value and therefore some part of saving is held in the form of monetary asset rather than in the form of real capital. However, Levhari and Patinkin correctly asserts that the equilibrium capital intensity can be higher in the monetary model if the physical savings ratio becomes higher than the overall savings ratio as a result of the introduction of money into the model.

A question of whether the equilibrium capital intensity is lower or higher in the monetary "economy" is fundamentally a question which is concerned with the development of financial institutions. As Tobin points out, an important question involved in this discussion is "the bearing of alternative financial policies and institutional arrangements on the supply of saving available for capital formation," i.e., a question of optimal degree of financial intermediation — "optimal in the sense that the technique used to link the savings of surplus units with the financial needs of deficit units in the maximum allocative efficiency at minimum inputs of real resources."[1] This is too tough a question to handle at the present stage of development of economic theory.[2]

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[1]Alvin L. Marty, "Notes on Money and Economic Growth," Journal of Money, Credit and Banking, I (May, 1969), p. 264.

[2]A pioneering attempt to discuss this problem may be found in Tobin, "Notes on Optimal Monetary Growth," pp. 843-59.

Summary

We have reviewed in this chapter the four representative monetary growth models with a special emphasis on the three central problems. The models of Tobin, Levhari-Patinkin, and Sidrauski are all based upon the simplifying assumption that there exist only two kinds of nonhuman wealth, namely real capital and outside money. It is assumed that the government injects outside money into the economy by means of transfer payments and withdraws it by means of taxes. Another characteristic common to all of these models is that they are all equilibrium models in the sense that it is assumed in these models that all markets are in equilibrium. On the other hand, Stein model is unique in that it is a disequilibrium model. Although it is assumed that the labor market is in equilibrium, both commodity and money markets are normally out of equilibrium. When there exists an excess demand in commodity market, prices rise. Postulating an independent investment function, the actual growth of the capital stock is assumed to be a linear combination of planned saving and planned investment instead of the saving function determining the capital formation. Unlike the models of Tobin, Levhari-Patinkin and Sidrauski, there is no rigid assumption that there are only two kinds of store of value and that money is all outside-type.

All the models presented so far, whether they are equilibrium or disequilibrium models, assume a conventional neoclassical production function homogeneous of degree one and well-behaved, except that in Levhari-Patinkin's producer's good approach real balances are treated as a factor of production function. However, the role of real balances is not the same in all models. Tobin and Stein fail to include the utility or productivity gain from real balances in the disposable income whereas in the models of Levhari-Patinkin and Sidrauski the utility or productivity gains from real balances are duely included in the disposable income. In all models the real balances are viewed as an asset or an alternative store of value.

Once we introduce an alternative store of value, money, into the model, the saving is no longer identical with the investment in real capital even in the equilibrium model and the real income consists not only of goods and services produced but also of the net change in the real value of money balances. Tobin demonstrates that the monetary policy can influence the equilibrium values of real variables even in the long run and hence the Harrodian impasse can be removed by appropriate monetary policies regulating the real rate of interest, $i - \pi$. Since the real rate of interest is equal to $i - \mu + n$ in the steady state, given the nominal rate of interest, i, the government can manipulate the real rate of interest by varying the rate of monetary expansion, μ, and thereby affect the equilibrium capital-output (and/or wealth-income) ratio and equilibrium capital-labor ratio. Given the nominal rate of interest, an increase in the rate of monetary expansion will lower the equilibrium wealth-income ratio but increase the equilibrium capital-output ratio thereby raising the equilibrium capital-labor ratio.

Tobin also shows that the equilibrium capital intensity is lower in the monetary model. This, of course, does not mean that the equilibrium capital intensity is lower in the monetary "economy".

As for the optimality question, Tobin maintains that any real rate of interest below the natural rate is not optimal because "all generations could have higher consumption by saving less and having a lower capital-output ratio."[1] When real rate of interest exceeds n, it is not optimal either to absorb any saving in government debt. Tobin's conclusion is that "there is an optimal rate of growth of the supply of outside money, equal to or smaller than the nominal interest rate, i, only to the extent that diversion of saving into this vehicle is necessary to keep the marginal productivity of capital from falling below n." However, this conclusion does not apply to the case of money as a means of payment instead of as a store of value. According to Tobin the optimum quantity of means of payment is probably "a large enough stock of means of payment to absorb all working balances," provided that there is no cost to society in creating means of payment. Of course, supplying a large enough stock of payment requires that means of payment bear a high enough real rate of return to remove the incentive to economize them.

Levhari and Patinkin demonstrate that the equilibrium capital intensity can be higher in the monetary model. Since the physical savings ratio is relevant in determining capital accumulation and since the physical savings ratio is not constant even when the overall savings ratio is constant, the equilibrium capital intensity in the monetary model will be higher or lower according as the physical savings ratio increases or falls as a result of the introduction of money into the model. This result is, of course, due to the fact that in Levhari-Patinkin model the utility or productivity gains from real balances i. included in the disposable income.

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[1]Tobin, Ibid., p. 838.

As far as the comparative-dynamic aspects of growth equilibrium in Levhari-Patinkin model is concerned, the effects of a change in the rate of monetary expansion on the equilibrium values of real variables are indeterminate. However, when money is considered as a consumer's good, a sufficiently large decline of the rate of monetary expansion must necessarily cause a decrease in the equilibrium capital intensity.

In Levhari-Patinkin model the optimal rate of monetary expansion is not in general the one that will guarantee equilibrium capital intensity specified by the Golden Rule and the optimum stock of money is not necessarily the satiety stock. Rather the optimal government policy is to keep the stock of money constant and let the general price level decrease at the natural rate.

Sidrauski model is based on the individual's utility-maximizing behavior and therefore is an optimum growth model. Sidrauski's conclusion on the comparative-dynamic question is that the long-run capital stock of money is independent of the rate of monetary expansion although in the short run an increase in the rate of monetary expansion reduces the rate of capital accumulation. This result is, of course, due to the assumption that the subjective rate of time preference is constant and unaffected by the stock and composition of nonhuman wealth.

Stein builds a disequilibrium model in the Keynes-Wicksell tradition and concludes that the equilibrium values of capital-labor ratio and per capita real balances can either fall, rise, or remain constant as a result of a rise in the rate of monetary expansion. Stein does not discuss the problem of optimum growth.

CHAPTER II

A MODEL ECONOMY AND ITS CIRCULAR FLOW

The primary objective of this chapter is to postulate a simple model economy capable of growing over time. The model economy to be presented in this chapter differs from others examined so far in several respects. First of all, unlike the conventional monetary models no distinction is made between the inside and outside money as a form of wealth. Secondly, both inside and outside money are considered as a factor of production. Thirdly, the bonds market is introduced and both the inside and outside bonds are treated as a factor of production as well as a form of wealth.[1]

Our model economy is a three-sector and four-market economy. There are consumers, business and government sectors. There are markets for commodities (goods and services), labor, money and bonds. Each sector engages in economic transactions in various markets determining the quantities supplied and demanded and hence the prices of commodities, labor service, money and bonds. However, it is assumed that the government does not buy or sell in the commodity market. The growth of the supply of labor is assumed to be exogenously determined.

The government creates and retires its debt, thereby regulating the nominal stock of its debt outstanding. Government debt is either interest-bearing or non-interest bearing. The latter is called "outside"

[1]Stein model also contains bonds market but only outside asset (money and bonds) are considered as a part of community's wealth.

money in the sense that it represents the private sector's net claim upon the government sector and is created on the basis of government holdings of gold or foreign securities. The former is "bond" which is outside-type in the same sense that the non-interest-bearing government debt is outside-type. The bonds are also issued by business firms, which are "inside" bonds in the sense that they represent the debts and claims within the private sector. The bonds, whether they are government bonds or private bonds, are of the same quality, that is, they are all guilt-edged perpetuities. However, there is no real difference between the government (outside) bonds and the business (inside) bonds as far as their wealth effect is concerned, as long as the business bonds remain as perpetuities. This point is discussed in detail later in this chapter. Thus the government can create "inside" money by purchasing business bonds or destroy it by selling them in the open market. The commercial banks can also create "inside" money in the form of demand deposits, which is considered a form of wealth. The "inside" money can be considered a form of wealth as long as, or to the extent that, the banks do not need to hold reserves to ensure confidence in its solvency as will be discussed later. Therefore, we have in the money market outside money created by government and inside money created either by government or by commercial banks. In the bond market we have also inside and outside bonds. Thus the nominal stock of money is the sum of nominal stock of inside and outside money. The nominal stock of government debt is the sum of nominal stock of outside money and government bonds. However, it is assumed that under the usual circumstances the government does not create inside money.

The aggregate production function of the economy is of the form

(2.1) $Y = Y(K,L,M/P,B/P)$

where Y = physical output (goods and services), K = capital, L = effective labor, M/P = real money balances and B/P = real bonds (real value of bonds). Real balances and real bonds are treated as factors of production for the following reasons: The introduction of money into the economy is equivalent to an increase in real resources which otherwise would have been tied up with barter transactions and can now be used in the production of goods and services. Similarly, the introduction of bonds is equivalent to an increase in real resources because it makes some extra amount of real resources available in the production process by facilitating financial intermediation between the savings of surplus units and the investments of the deficit units.[1] The only difference between money and bonds is that the former is non-interest-bearing because it is highly liquid and involves little uncertainty, and the latter is interest-bearing because it is less liquid and involves greater degree of uncertainty than the former. However, the real rates of return on both types of assets must be equal in equilibrium. In other words, money and bond are introduced into the economy because they are useful, i.e., productive. The existence of the positive real rates of return on these monetary assets proves this

[1]Money and bonds are factors of production in as much as the introduction of these assets represents the release of real resources tied up with barter transactions or an increase in the efficiency of the economy which may be measured by the increase in the productivity of real capital in the conventional methods. The increase in the productivity of real capital made possible by the introduction of monetary assets are to be attributed to these monetary assets.

fact. The production function represented by (2.1) is assumed to be homogeneous of degree one in each argument and well-behaved.[1] Hence it may be rewritten as

(2.2) $y = y(k,m,b)$

where $y = Y/L$, $k = K/L$, $m = M/(PL)$ and $b = B/(PL)$.

In a monetary economy the real disposable income of the economy consists not only of physical output, Y, but also of the net increase in the real value of monetary assets. Hence the real disposable income of the economy, Y*, is

(2.3) $Y^* = Y + \dot{M}/P + \dot{B}/P$

where M and B stand for the nominal stock of money and bonds respectively and the dot $(\cdot)$ represents the operator, d/dt.[2] An important point to be emphasized here is that both inside and outside money are treated as a part of community's wealth and likewise for bonds.

Gurley and Shaw introduced the idea that outside money can properly be considered as an addition to wealth of the economy but inside money, being able to be cancelled out within the private sector as debts and claims, can by no means be so considered.[3] Pesek and Saving reject this

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[1]The economists who have been uncomfortable with the assumption of variable factor proportions have tried to introduce different assumptions. The Kaldorian concept of factor specificity in the vintage model, the Schumpeterian technological shifts of the production function and the putty-clay assumption of Phelps are the well-known examples.

[2]It is assumed throughout this chapter that the general price level does not change, for the sake of simplicity.

[3]J. G. Gurley and E. S. Shaw, Money in a Theory of Finance (Washington: The Brookings Institution, 1960).

distinction between inside and outside money and correctly assert that both types of money constitute net wealth for the community.[1] As long as solvency of the banking system is preserved, the two types of money are exactly equivalent in their effects on society's wealth. As long as debts and claims remain as they are without being physically cancelled out and as long as no real capital (resource) is required to create them, there is no real distinction between inside and outside money or between inside and outside bonds. (Remember the assumption that the bonds, whether they are issued by the government or by the business firms, are all guilt-edged perpetuities.) This is true because neither the commercial banks will expect their deposits to be withdrawn immediately nor will the corporations expect their bonds outstanding to be claimed immediately. Even if the inside bonds are not perpetuities, they are an addition to the community's wealth to the extent that the corporations expect a stream of future incomes from the investment financed by the bonds. As long as the confidence prevails between lenders and borrowers, both inside and outside assets, whether money or bonds, i.e., whether interest-bearing or not, will add to the community's wealth. The best way to understand this point is to assume that the society is initially under the commodity money conditions and then to see how the wealth of the society changes when the commodity money is replaced by outside credit money and with inside credit money.[2] The replacement of the commodity money by credit

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[1]B. P. Pesek and T. R. Saving, Money, Wealth and Economic Theory (New York: Macmillan, 1967).

[2]This method is used in H. G. Johnson, "Inside Money, Outside Money, Income, Wealth and Welfare in Monetary Theory," pp. 34-5.

money can take place in two ways. First, the government can issue certificates in return for the commodity money, the monetary value of the certificates being ensured by the government. In this case, to the extent that the government does not need to hold reserves of commodity money to ensure the confidence in its certificates, the physical resources embodied in the commodity money could be reallocated into the production of goods and services. The government certificates may be called "outside" money and add to the wealth of the community, since the physical resources tied up with the commodity money can now be used in the production of goods and services. Second, the commercial banks can issue demand deposits in return for the commodity money. In this case, to the extent that the banks do not need to hold reserves of commodity money to ensure confidence in its solvency, the physical resources embodied in commodity money could be reallocated into the production of goods and services. The demand deposits may be called "inside" money and add to the wealth of the community since the physical resources embodied in the commodity money can now be used in the production of goods and services.

The similar argument can be made for the case of inside and outside bonds. The "outside" bonds are created by means of government borrowing. To the extent that the government does not need to hold reserves to ensure confidence in its solvency, the outside bonds can be considered as adding to the wealth of the community since the fund raised by issuing bonds can be used in the production of goods and services. The "inside" bonds are created by means of corporate borrowing. To the extent that the corporations do not need to hold reserves to ensure confidence in their solvency, the inside bonds add to the wealth of the community since the fund raised by issuing bonds can be used in the production of goods and services.[1] The corporate bonds add to the external source of total corporate saving without decreasing the wealth of the bond-holders as long as no resources are required to support the soundness of their bonds. After all, the only difference between money and bonds (and hence inside money and inside bonds) is that the latter is interest-bearing whereas the former is not, this fact being attributable to the differences in liquidity and risk involved in these two types of monetary assets.

A part of real disposable income of the nation is consumed and the rest is saved. In a monetary economy saving is held either in the form of real capital or in the form of monetary assets. Thus, real saving of the economy, S, is

$$(2.4) \quad S = \dot{K} + \dot{M}/P + \dot{B}/P.$$

Equation (2.4) represents a fundamental departure from the conventional non-monetary model.

The model economy grows over time because of two reasons. First, its real disposable income grows because of growth of physical output

[1]For simplicity, it is assumed that the bonds, inside or outside, are all guilt-edged perpetuities. However, the fact that inside bonds are a form of wealth does not depend upon this assumption. Although the corporations may regard debt as a constraint in their investment decisions and related managerial decisions, the inside bonds can be regarded as a form of wealth to the extent that the investment financed by the bonds generates a stream of future incomes which otherwise would not be generated. In fact, any form of debt, short- or long-term, can be considered as adding to the society's wealth to some extent. After all, why do people borrow and lend if the borrowing and lending do not increase the welfare of the society?

(goods and services), Y. Second, its real disposable income grows because of growth of real stock of monetary assets, M/P + B/P. The physical output grows over time as the factors of production, K,L,M/P and B/P, grow and technology advances. Real stock of monetary assets grows over time when the nominal stock of monetary assets grows and when prices fall. Therefore, given the initial values of K,L,M,B and P, and assuming that the rate of growth of population and the rate of technological progress are exogenously determined, the actual growth of the economy will be determined by the saving behavior and the growth of monetary assets as well as the rate of price change. In other words, the dynamic system boils down to a set of differential equations which describes the time profiles of endogenous real variables, k, m and b.

In Chapter III a dynamic system of our model economy will be developed and the comparative-dynamic properties of its growth equilibrium will be discussed. The optimality question will be taken up in Chapter V. The basic approach to be used in Chapter III is one of equilibrium model building.

CHAPTER III

THE COMPARATIVE-DYNAMIC ASPECTS OF GROWTH EQUILIBRIUM I: AN EQUILIBRIUM MODEL

In Chapter II a model economy whose production function is of the form $Y = Y(K,L,M/P,B/P)$ and in which four markets exist was introduced. In this chapter we are primarily concerned with the comparative-dynamic properties of growth equilibrium in this model economy. Thus an important question to be asked in this chapter is: What are the effects of a change in the rate of monetary expansion on the equilibrium capital intensity and per capita real balances, and hence on the real rental rate and real wage rate?

The model to be presented in this chapter is very similar to the Levhari-Patinkin model. However, there are three important departures from the latter. First, even though both models include real balances in the production function as a factor of production, the implications are not the same in both models. In the Levhari-Patinkin model it is assumed that "all money balances are held by the business sector of the economy"[1] and the real balances are treated as a factor of production because "the entrance of money into the production function reflects the fact that it frees labor and capital for the production of commodities proper."[2] Hence

[1]Levhari and Patinkin, op. cit., p. 737.

[2]Ibid., p. 738.

the production function is the business sector's production function. However, it is assumed in this chapter that the real balances are held both by the business firms and by the consumers and that the total stock of money existing in the economy enters the production function increasing the overall productivity of the entire economy. Therefore, the production function represented by (2.1) is the economy's aggregate production function.

Second, the bond market is introduced in this model while Levhari and Patinkin make a simplifying assumption that outside money is the only financial asset existing in the economy. The bonds are also considered a factor of production as well as a form of wealth. Stein model includes the bonds market. However, Stein treats bonds in the same way as he does money despite the fact that their roles in the economy are quite different. The bonds cannot serve the function of medium of exchange or means of payment adequately. The bonds bear higher nominal rate of return than money because the bonds are not as liquid as money and are a riskier asset than money. Real capital bears the highest nominal rate of return of all forms of nonhuman assets. In fact, the most important factors that differentiate between real capital, bond and money seem to be the liquidity and the amount risk involved, money being the most liquid and involving the least amount of risk and real capital being the least liquid and involving the greatest amount of risk. This seems to be the reason why the nominal rate of return is the highest on real capital and the lowest on money. Stein simplifies the case by including bonds in the real balances. His "rate of monetary expansion" is the rate of increase in the nominal stock of outside money and outside bonds.

Finally, it is assumed in this model that the inside money and inside bonds are a form of community's asset as well as outside money and outside bonds. The validity of this assumption is discussed in Chapter II.

The model is an equilibrium model in the sense that all four markets are assumed to be in equilibrium and hence we ignore the problem of short-run adjustments but concern ourselves exclusively with the state of equilibrium. Hence, there is no independent investment function. Once the saving hypothesis is chosen, it determines the path of capital accumulation, together with the rate of price change which will again be determined by the rate of monetary expansion.

We start with the production function introduced in Chapter II which is written in per capita terms as follows:

$$(3.1) \quad y = y(k,m,b)$$

where y_k, y_m, $y_b > 0$; y_{kk}, y_{mm}, $y_{bb} < 0$; and y_{km}, y_{kb}, $y_{mb} > 0$ due to the assumption of well-behavedness of the production function. We assume that the labor force measured in efficiency units (effective labor) grows at the exogenously determined rate, n. The rate of growth of effective labor is the Harrodian natural rate of growth which is the sum of the rate of growth of population and the rate of technological progress.

In our monetary economy the real disposable income of the nation is not just the physical output (goods and services) but it also includes the increase in the real stock of money and bonds. Hence the real disposable income, Y*, is

$$(3.2) \quad Y^* = Y + D(M/P) + D(B/P) = Y + (\mu - \pi)M/P + (\gamma - \pi)B/P$$

where μ and γ stand for the rates of growth of nominal stock of money and bonds respectively. The real disposable income defined in (3.2) differs from Levhari-Patinkin's (producer's good approach) and also from Tobin's definition not only because it includes the increase in real stock of bonds but also because money and bonds include inside money and inside bonds respectively.

We assume that a constant proportion, s, of the real disposable income rather than physical output is saved for the purpose of capital as well as wealth accumulation, so that

(3.3) $S = sY^*.$

It follows from (2.4) in Chapter II that

(3.4) $S = DK + D(M/P) + D(B/P)$

where DK stands for the accumulation of real capital and D(M/P) + D(B/P) stands for the accumulation of monetary assets. What is implied in (3.4) is that the savers hold their savings either in the form of real capital or in the form of monetary assets, or both.

Combining equations (3.1) - (3.4) together yields a differential equation[1]

(3.5) $\dot{k} = sy(k,m,b) - (1 - s)[(\mu - \pi)m + (\gamma - \pi)b] - nk$

which describes the time profile of the capital-labor ratio (capital intensity) in our monetary model. It is clear from (3.5) that the time path of the capital intensity is determined not only by the propensity to

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[1] For mathematical derivations, see Appendix VI in which a formal equilibrium model is developed for this chapter.

save of the society but also by the rate of monetary expansion. This implies that monetary policy will affect the real variables. The variations in the rate of monetary expansion will affect the real variables in two ways. They change the level of physical output via production function on the one hand and change the real stock of wealth on the other hand. The changes in the level of physical output and real stock of wealth imply changes in the rate of capital accumulation and, therefore, changes in the capital-labor ratio, real rental rate and real wage rate.

Since there are three endogenous variables, k, m, and b in the system, which, given parameters, μ, γ, π, n and s, determine real disposable income, saving and accumulation of wealth, a complete dynamic system requires three differential equations describing the time profiles of all three variables, k, m and b. To obtain differential equations describing time paths of m and b, we differentiate $m = M/(PL)$ and $b = B/(PL)$ with respect to time, after transforming them into logarithmic expressions. Then we have

(3.6) $\dot{m} = (\mu - \pi - n)m$

(3.7) $\dot{b} = (\gamma - \pi - n)b.$

A system of differential equations (3.5) - (3.7) constitutes a dynamic system of our model economy.

From (3.5) - (3.7) we can derive the steady-state conditions. Since in the steady state (growth equilibrium) all real variables grow at the same rate, the steady-state conditions can be obtained by setting $\dot{k}$, $\dot{m}$ and $\dot{b}$ all equal to zero. Hence, the steady-state conditions are